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On $J(R)$ of the Semilocal Rings

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Abstract

Given a semilocal commutative ring with unity R, I give a necessary and sufficient condition to $J(R)$ in order to assert that R is either a noetherian ring or an artinian ring.

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1 Introduction

In this paper it is given a new characterization of two particular families of semilocal commutative rings: the artinian rings and the quasi-semilocal rings. Both of these characterizations focus on the analysis of the ideals of the ring inside the Jacobson radical. Hence the aim of this paper is to find simpler conditions to determine if a semilocal commutative ring with unity is either artinian or noetherian.

There is a large number of characterizations of more generic semilocal rings in literature. Some of them, for example, are collected in [1], [2], [8] and [9].

In the present paper the characterizations are based on the two main theorems of section 5.

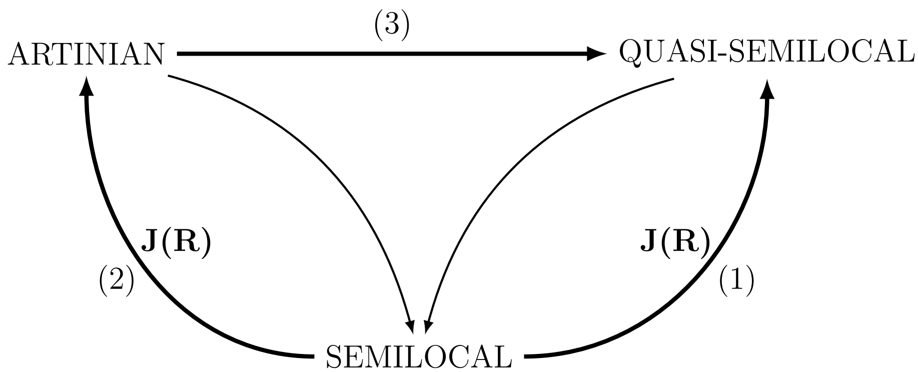
In regard to the quasi-semilocal rings, Theorem 5.1 states:
"Let R be a semilocal commutative ring with unity. Then R is noetherian if and only if any I ideal of R such that $I \subseteq J(R)$ is finitely generated."

In regard to the artinian rings, Theorem 5.2 states:
"Let R be a commutative ring with unity. Then R is artinian if and only if

R is a semilocal ring in which every descending chain of ideals I_j of R, with $I_j \subseteq J(R)$, becomes stationary."

Both of the proofs use just basic notions of a commutative algebra course, thus making the paper suitable for didactic purposes as well.

The following graph summarizes the previous theorems.



(1) Theorem 5.1; (2) Theorem 5.2; (3) Akizuki-Hopkins-Levitzki Theorem. The other two arrows are trivial.

2 Notation

Throughout, let R be an associative ring.

Definition 2.1. *I is a maximal (left) ideal of R if for any K (left) ideal of R and $I \subseteq K$, then either $K = I$ or $K = R$.*

Definition 2.2. *The Jacobson radical of R is the intersection of all maximal (left) ideals of R. It is denoted by $J(R)$.*

Definition 2.3. *A ring R is called Jacobson semisimple (or J-semisimple) if $J(R) = (0)$.* [6]

Definition 2.4. *Let R be a commutative ring with unity, then R is semilocal if it has finitely many maximal ideals.*

Definition 2.5. *Let R be a commutative ring with unity, then R is quasi-semilocal if it is semilocal and noetherian.*

I mention a definition and Theorem 3.4.1 from [5]:

Definition 2.6. *"A mapping Φ from the ring R into the ring R' is said to be a homomorphism if $\Phi(a + b) = \Phi(a) + \Phi(b)$ and $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in R$."*

Theorem 2.7. *"Let R, R' be rings and Φ a homomorphism of R onto R' with kernel U. Then R' is isomorphic to R/U. Moreover there is a one-to-one correspondence between the set of ideals of R' and the set of ideals of R which contain U. This correspondence can be achieved by associating with an ideal W' in R' the ideal W in R defined by $W = \{x \in R \mid \Phi(x) \in W'\}$. With W so defined, R/W is isomorphic to R'/W'."*

3 Preliminary lemmas

Lemma 3.1. *Let R be a commutative ring; let M be a ideal of R. If R/M is a field then M is a maximal ideal of R.*

Proof. If we suppose that M is not a maximal ideal of R, then there exists an ideal N of R such that $M \subsetneq N \subsetneq R$. Let π be the canonical projection $\pi : R \to R/M$. Then, by Theorem 2.7, $\pi(M) \subsetneq \pi(N) \subsetneq \pi(R)$. Hence $\pi(N) = N/M$ is a non-trivial ideal of R/M and R/M is not a field. $\square$

Lemma 3.2. *Let R be a commmutative ring with unity; let M be a maximal ideal of R; let I be an ideal of R such that $I \nsubseteq M$. Then $I/(M \cap I)$ is a field and $M \cap I$ is a maximal ideal of I.*

Proof. Since R is a commutative ring with unity and M is a maximal ideal of R, R/M is a field. Since $I \nsubseteq M$, $R = I + M$. We define Φ as:

$$\Phi : I \to R/M \qquad\qquad \Phi(i) := i + M.$$

1. Φ is a homomorphism:
$$\Phi(i_1 + i_2) = i_1 + i_2 + M = (i_1 + M) + (i_2 + M) = \Phi(i_1) + \Phi(i_2)$$
$$\Phi(i_1 i_2) = i_1 i_2 + M = (i_1 + M)(i_2 + M) = \Phi(i_1)\Phi(i_2).$$
2. $Ker(\Phi) = M \cap I$.
3. Φ is surjective: since $R = I + M$, there exists $i \in I$ and $m \in M$ such that $r = i + m$. If $r + M \in R/M$, then
$$r + M = i + m + M = i + M = \Phi(i).$$

Hence
$$I/(M \cap I) \cong R/M.$$
$I/(M \cap I)$ is a field and, by Lemma 3.1, $M \cap I$ is a maximal ideal of I. $\square$

4 Lemmas of the minimal family of subsets

Lemma 4.1. *Let A be a set and let $\mathcal{F} \neq \emptyset$ be a finite family of subsets of A such that $\bigcap_{S\in\mathcal{F}} S = I$. Then there exists a minimal family $\bar{\mathcal{F}} \subseteq \mathcal{F}$, $\bar{\mathcal{F}} \neq \emptyset$, such that $\bigcap_{S\in\bar{\mathcal{F}}} S = I$ and, for any $\bar{S} \in \bar{\mathcal{F}}$, $\bigcap_{S\in\bar{\mathcal{F}}\setminus\{\bar{S}\}} S \neq I$.*

Proof. By induction on the cardinality of $\mathcal{F}$.
Base case: if $\mid \mathcal{F} \mid= 1$, then $\mathcal{F} = \{S_1\}$ and $\bar{\mathcal{F}} = \mathcal{F} \neq \emptyset$.
Inductive step: if we suppose the lemma true for the families with cardinality less than $\mid \mathcal{F} \mid$, then we can prove it for $\mathcal{F}$.
If we define $\mathcal{F} := \{S_1, ..., S_n\}$, then in $\mathcal{F}_1 := \mathcal{F} \setminus \{S_n\}$, by inductive hypothesis, there exists a minimal family $\bar{\mathcal{F}}_1$, $\bar{\mathcal{F}}_1 \subseteq \mathcal{F}_1$, such that:

$$\bigcap_{S\in\bar{\mathcal{F}}_1} S = \bigcap_{S_i\in\mathcal{F}_1} S_i \text{ and, } \forall\bar{S} \in \bar{\mathcal{F}}_1, \bigcap_{S\in\bar{\mathcal{F}}_1\setminus\{\bar{S}\}} S \neq \bigcap_{S_i\in\mathcal{F}_1} S_i.$$

Since

$$\bigcap_{S_i\in\mathcal{F}} S_i = \bigcap_{S_i\in\mathcal{F}_1\cup\{S_n\}} S_i = (\bigcap_{S_i\in\mathcal{F}_1} S_i) \cap S_n = (\bigcap_{S\in\bar{\mathcal{F}}_1} S) \cap S_n = \bigcap_{S\in\bar{\mathcal{F}}_1\cup\{S_n\}} S,$$

$$\text{if} \bigcap_{S\in\bar{\mathcal{F}}_1\cup\{S_n\}} S = \bigcap_{S\in\bar{\mathcal{F}}_1} S, \text{ then } \bar{\mathcal{F}} = \bar{\mathcal{F}}_1; \qquad \text{if} \bigcap_{S\in\bar{\mathcal{F}}_1\cup\{S_n\}} S \neq \bigcap_{S\in\bar{\mathcal{F}}_1} S,$$

then there are two cases:

$$\text{if } \forall\bar{S} \in \bar{\mathcal{F}}_1 \bigcap_{S\in(\bar{\mathcal{F}}_1\cup\{S_n\})\setminus\{\bar{S}\}} S \neq \bigcap_{S\in\bar{\mathcal{F}}_1\cup\{S_n\}} S, \text{ then } \bar{\mathcal{F}} = \bar{\mathcal{F}}_1 \cup \{S_n\};$$

$$\text{if } \exists\bar{S} \in \bar{\mathcal{F}}_1, \text{ such that} \bigcap_{S\in(\bar{\mathcal{F}}_1\cup\{S_n\})\setminus\{\bar{S}\}} S = \bigcap_{S\in\bar{\mathcal{F}}_1\cup\{S_n\}} S,$$

then we define $\mathcal{F}_2 := (\bar{\mathcal{F}}_1 \cup \{S_n\}) \setminus \{\bar{S}\}$. By inductive hypothesis, there exists a minimal family $\bar{\mathcal{F}}_2$ and $\bar{\mathcal{F}} = \bar{\mathcal{F}}_2$. $\bar{\mathcal{F}} \neq \emptyset$ in any case. □

Lemma 4.2. *Let A be a set; let $\mathcal{F} \neq \emptyset$ be a finite minimal family of subsets of A with intersection I (hence such that $\bigcap_{S\in\mathcal{F}} S = I$ and, $\forall\hat{S} \in \mathcal{F}$, $\bigcap_{S\in\mathcal{F}\setminus\{\hat{S}\}} S \neq I$). Then, for any $\mathcal{F}_1 \subsetneq \mathcal{F}$ and for any $\bar{S} \in \mathcal{F} \setminus \mathcal{F}_1$, $\bar{S} \cap (\bigcap_{S\in\mathcal{F}_1} S) \neq \bigcap_{S\in\mathcal{F}_1} S$.*

Proof. By absurd, if we suppose that there exists $\bar{S} \in \mathcal{F} \setminus \mathcal{F}_1$ such that:

$$\bigcap_{S\in\mathcal{F}_1\cup\{\bar{S}\}} S = \bar{S} \cap (\bigcap_{S\in\mathcal{F}_1} S) = \bigcap_{S\in\mathcal{F}_1} S,$$

$$\text{then } I = \bigcap_{S\in\mathcal{F}} S = (\bigcap_{S\in\mathcal{F}\setminus(\mathcal{F}_1\cup\{\bar{S}\})} S) \cap (\bigcap_{S\in\mathcal{F}_1\cup\{\bar{S}\}} S) =$$

$$= (\bigcap_{S\in\mathcal{F}\setminus(\mathcal{F}_1\cup\{\bar{S}\})} S) \cap (\bigcap_{S\in\mathcal{F}_1} S) = \bigcap_{S\in\mathcal{F}\setminus\{\bar{S}\}} S \neq I. \text{ Absurd. } \square$$

5 Main results

Theorem 5.1. *Let R be a semilocal commutative ring with unity. Then the following statements are equivalent:*
(a) R is noetherian (quasi-semilocal);
(b) for any I ideal of R and $I \subseteq J(R)$, I is finitely generated.

Proof. $(a) \Rightarrow (b)$ If I is an ideal of R, $I \subseteq J(R)$, and I is not finitely generated, then R is not a noetherian ring.

$(b) \Rightarrow (a)$ R is a commutative ring with unity, hence $R = 1 * R$. Let I be an ideal of R, if $I \subseteq J(R)$, then, by hypothesis, I is finitely generated. Let us assume that $I \nsubseteq J(R) := \bigcap_{j=1}^{n} M_j$, with the M_j maximal ideals of R. Hence, there exists $\bar{M}$ maximal ideal of R, such that $\bar{M} \cap I \neq I$. We define:
$A := \{M_j \mid M_j \text{ is a maximal ideal of } R\}$,
$B := \{M_j \in A \mid M_j \cap I \neq I\}$,
$\mathcal{F} := \{M_j \cap I \mid M_j \in B\}$.
$\bar{M} \in B$, hence $\bar{M} \cap I \in \mathcal{F}$ and $\mathcal{F} \neq \emptyset$.
If $M_j \in A \setminus B$, then $M_j \supseteq I$, $\bigcap_{M_j \in A\setminus B} M_j \supseteq I$ and $I = I \cap (\bigcap_{M_j \in A\setminus B} M_j)$.

$$\bigcap_{M_j \cap I \in \mathcal{F}} (M_j \cap I) = I \cap (\bigcap_{M_j \in B} M_j) = I \cap (\bigcap_{M_j \in A\setminus B} M_j) \cap (\bigcap_{M_j \in B} M_j) =$$

$$= I \cap (\bigcap_{M_j \in A} M_j) = I \cap J(R)$$

By Lemma 4.1, there exists a minimal family $\bar{\mathcal{F}} \subseteq \mathcal{F}$ and $\bar{\mathcal{F}} \neq \emptyset$, such that:

$$\bigcap_{\bar{M}_j \cap I \in \bar{\mathcal{F}}} (\bar{M}_j \cap I) = I \cap J(R) \text{ and } \forall \bar{M}_x \cap I \in \bar{F} \bigcap_{\bar{M}_j \cap I \in \bar{\mathcal{F}} \setminus \{\bar{\mathcal{M}}_x \cap I\}} (\bar{M}_j \cap I) \neq I \cap J(R).$$

$\mid \bar{\mathcal{F}} \mid = m$, with $1 \leq m \leq n = \mid \mathcal{F} \mid$, and $\bar{\mathcal{F}} := \{\bar{M}_1 \cap I, ..., \bar{M}_m \cap I\}$, with $\bar{M}_j \in B$.
Since $\bar{\mathcal{F}} \neq \emptyset$, $\bar{M}_1 \cap I \in \bar{\mathcal{F}}$ and $\bar{M}_1 \cap I \neq I$, hence $R = \bar{M}_1 + I$, because $\bar{M}_1$ is a maximal ideal of R. By Lemma 3.2, $I/(\bar{M}_1 \cap I)$ is a field and $\bar{M}_1 \cap I = I\bar{M}_1$ is a maximal ideal of I. Hence $I = i_0 R + I\bar{M}_1$, with $i_0 \in I \setminus I\bar{M}_1$.
If $\mid \bar{\mathcal{F}} \mid = 1$, then $I\bar{M}_1 = I \cap \bar{M}_1 = I \cap J(R)$ and $I = i_0 R + I\bar{M}_1 = i_0 R + I \cap J(R)$.
If $\mid \bar{\mathcal{F}} \mid > 1$, then there exists $\bar{M}_2 \cap I \in \bar{\mathcal{F}}$, with $\bar{M}_2 \in B$.
$\bar{M}_2 \cap (I \cap \bar{M}_1) = \bar{M}_2 \cap I \cap \bar{M}_1 = (\bar{M}_2 \cap I) \cap (I \cap \bar{M}_1) \neq \bar{M}_1 \cap I$, by Lemma 4.2. Hence $R = \bar{M}_2 + I\bar{M}_1$ and, by Lemma 3.2, $I\bar{M}_1/(I\bar{M}_1 \cap \bar{M}_2)$ is a field and $I\bar{M}_1 \cap \bar{M}_2 = I\bar{M}_1\bar{M}_2$ is a maximal ideal of $I\bar{M}_1$. Hence there exists $i_1 \in I\bar{M}_1 \setminus I\bar{M}_1\bar{M}_2$ such that $I\bar{M}_1 = i_1 R + I\bar{M}_1\bar{M}_2$.
$I = i_0 R + I\bar{M}_1 = i_0 R + i_1 R + I\bar{M}_1\bar{M}_2$.
$(\bar{M}_1 \cap I) \cap (\bar{M}_2 \cap I) = (\bar{M}_1 \cap I) \cap \bar{M}_2 = (I\bar{M}_1) \cap \bar{M}_2 = I\bar{M}_1\bar{M}_2$.
If $\mid \bar{\mathcal{F}} \mid = 2$, then $I\bar{M}_1\bar{M}_2 = (\bar{M}_1 \cap I) \cap (\bar{M}_2 \cap I) = I \cap J(R)$ and

$I = i_0R + i_1R + I\bar{M}_1\bar{M}_2 = i_0R + i_1R + I \cap J(R)$.
Inductive step: if $\mid \bar{\mathcal{F}} \mid > 2$, by using previous cases as base cases, we prove that:
"If there exists $i_0 \in I \setminus I\bar{M}_1$ and $i_h \in I\prod_{j=1}^{h} \bar{M}_j \setminus I\prod_{j=1}^{h+1} \bar{M}_j$ for all $1 \leq h \leq m-2$ such that $I = \sum_{h=0}^{m-2} i_hR + I\prod_{h=1}^{m-1} \bar{M}_h$ and $\bigcap_{h=1}^{m-1}(\bar{M}_h \cap I) = I\prod_{h=1}^{m-1} \bar{M}_h$, then there exists $i_{m-1} \in I\prod_{j=1}^{m-1} \bar{M}_j \setminus I\prod_{j=1}^{m} \bar{M}_j$ such that
$I = \sum_{h=0}^{m-1} i_hR + I\prod_{h=1}^{m} \bar{M}_h$ and $\bigcap_{h=1}^{m}(\bar{M}_h \cap I) = I\prod_{h=1}^{m} \bar{M}_h$."
Proof of the inductive step:

$$\bar{M}_m \cap (\bigcap_{h=1}^{m-1}(\bar{M}_h \cap I)) = \bar{M}_m \cap I \cap (\bigcap_{h=1}^{m-1} \bar{M}_h) = (\bar{M}_m \cap I) \cap (\bigcap_{h=1}^{m-1}(\bar{M}_h \cap I)) =$$

$$= \bigcap_{h=1}^{m}(\bar{M}_h \cap I) = \bigcap_{\bar{M}_h \cap I \in \bar{\mathcal{F}}}(\bar{M}_h \cap I) \neq \bigcap_{\bar{M}_h \cap I \in \bar{\mathcal{F}} \setminus \{\bar{M}_m \cap I\}}(\bar{M}_h \cap I) = \bigcap_{h=1}^{m-1}(\bar{M}_h \cap I),$$

because $\bar{\mathcal{F}}$ is a minimal family. Since $\bar{M}_m$ is a maximal ideal of R,
$R = \bar{M}_m + \bigcap_{h=1}^{m-1}(\bar{M}_h \cap I) = \bar{M}_m + I\prod_{h=1}^{m-1} \bar{M}_h$.
Hence

$$\bar{M}_m \cap (\bigcap_{h=1}^{m-1}(\bar{M}_h \cap I)) = \bar{M}_m \cap I\prod_{h=1}^{m-1} \bar{M}_h = I\prod_{h=1}^{m} \bar{M}_h$$

and

$$I \cap J(R) = \bigcap_{\bar{M}_h \cap I \in \bar{\mathcal{F}}}(\bar{M}_h \cap I) = \bigcap_{h=1}^{m}(\bar{M}_h \cap I) = \bar{M}_m \cap (\bigcap_{h=1}^{m-1}(\bar{M}_h \cap I)) = I\prod_{h=1}^{m} \bar{M}_h.$$

By Lemma 3.2, $I\prod_{h=1}^{m-1} \bar{M}_h/((I\prod_{h=1}^{m-1} \bar{M}_h) \cap \bar{M}_m)$ is a field and
$((I\prod_{h=1}^{m-1} \bar{M}_h) \cap \bar{M}_m) = I\prod_{h=1}^{m} \bar{M}_h$ is a maximal ideal of $I\prod_{h=1}^{m-1} \bar{M}_h$.
Hence there exists $i_{m-1} \in I\prod_{h=1}^{m-1} \bar{M}_h \setminus I\prod_{h=1}^{m} \bar{M}_h$ such that
$I\prod_{h=1}^{m-1} \bar{M}_h = i_{m-1}R + I\prod_{h=1}^{m} \bar{M}_h$. By inductive hypothesis,

$$I = \sum_{h=0}^{m-2} i_hR + I\prod_{h=1}^{m-1} \bar{M}_h = \sum_{h=0}^{m-2} i_hR + i_{m-1}R + I\prod_{h=1}^{m} \bar{M}_h = \sum_{h=0}^{m-1} i_hR + I\prod_{h=1}^{m} \bar{M}_h,$$

with $i_0 \in I \setminus I\bar{M}_1$ and $i_h \in I\prod_{j=1}^{h} \bar{M}_j \setminus I\prod_{j=1}^{h+1} \bar{M}_j$ for all $1 \leq h \leq m-1$.
This concludes the proof of the inductive step. Now,

$$I = \sum_{h=0}^{m-1} i_hR + I\prod_{h=1}^{m} \bar{M}_h = \sum_{h=0}^{m-1} i_hR + I \cap J(R), \text{ with } i_h \in I \text{ for all } 0 \leq h \leq m-1.$$

Since $I \cap J(R)$ is an ideal of R contained in $J(R)$, $I \cap J(R)$ is finitely generated by hypothesis: $I \cap J(R) = \sum_{i=0}^{s} a_iR$.

$$I = \sum_{h=0}^{m-1} i_hR + I \cap J(R) = \sum_{h=0}^{m-1} i_hR + \sum_{i=0}^{s} a_iR,$$

I is finitely generated, R is a noetherian ring. $\square$

Theorem 5.2. *Let R be a commutative ring with unity. Then the following statements are equivalent:*
(a) *R is artinian;*
(b) *R is a semilocal ring in which every descending chain of ideals I_j of R, with $I_j \subseteq J(R)$, becomes stationary.*

Proof. $(a) \Rightarrow (b)$ If R is artinian, then R is a semilocal ring. If there exists a descending chain of ideals I_j of R, with $I_j \subseteq J(R)$, which does not become stationary, then R is not an artinian ring.

$(b) \Rightarrow (a)$ By absurd, if we suppose that R is not an artinian ring, then there exists a descending chain of ideals, $I_0 \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq ....$, which does not became stationary. Hence we can extract from it a infinite strictly decreasing subchain of ideals $\bar{I}_j$ of R, $\bar{I}_0 \supset \bar{I}_1 \supset \bar{I}_2 \supset \bar{I}_3 \supset ....$ with $\bar{I}_0 := I_0$. For all j, $\bar{I}_j \nsubseteq J(R)$, because otherwise this chain would become stationary. Let $A := \{M_1, ..., M_n\}$ be the set of all maximal ideals of R for which there exists $\bar{j}$ such that $\bar{I}_{\bar{j}} \subseteq M_i$. By Krull's Theorem [7], $A \neq \emptyset$. Not all maximal ideals of R are contained in A, because otherwise there would exist an ideal $\bar{I}_{\bar{s}}$ such that $\bar{I}_{\bar{s}} \subseteq M$ for all maximal ideals M of R and hence $\bar{I}_{\bar{s}} \subseteq J(R)$. (If R is a local ring the proof is achieved.) Let $\bar{I}_{\bar{n}}$ be the biggest ideal of the chain (the minimum value of $\bar{n}$) such that $\bar{I}_{\bar{n}} \subseteq M_i$ $\forall 1 \leq i \leq n$. Let $M_{n+1}, ..., M_m$ be the other maximal ideals of R not contained in A. Then:

$$\forall\ \bar{I}_k \text{ with } k \geq \bar{n}: \quad \bar{I}_k \subseteq \prod_{i=1}^{n} M_i = \bigcap_{i=1}^{n} M_i, \quad \bar{I}_k \nsubseteq M_{n+1}, \quad \bar{I}_k \nsubseteq \prod_{i=1}^{n+1} M_i = \bigcap_{i=1}^{n+1} M_i.$$

Since the maximal ideals are prime ideals in R, $\prod_{i=1}^{n+1} M_i \neq \prod_{i=1}^{n} M_i$. By Lemma 3.2, $\prod_{i=1}^{n+1} M_i$ is a maximal ideal of $\prod_{i=1}^{n} M_i$, hence

$$\prod_{i=1}^{n} M_i = \bar{I}_k + \prod_{i=1}^{n+1} M_i \quad \forall k \geq \bar{n}.$$

$$\prod_{i=1}^{n} M_i = \bar{I}_{\bar{n}} + \prod_{i=1}^{n+1} M_i = \bar{I}_{\bar{n}+1} + \prod_{i=1}^{n+1} M_i = \bar{I}_{\bar{n}+2} + \prod_{i=1}^{n+1} M_i = \bar{I}_{\bar{n}+3} + \prod_{i=1}^{n+1} M_i = ...$$

$$\text{If } k \geq \bar{n}, \quad \bar{I}_k + \prod_{i=1}^{n+1} M_i = \prod_{i=1}^{n} M_i = \bar{I}_{k+1} + \prod_{i=1}^{n+1} M_i.$$

$$\text{If } i_k \in \bar{I}_k,\ i_k \in \bar{I}_{k+1} + \prod_{i=1}^{n+1} M_i,\ i_k = i_{k+1} + m \ \text{ with } \ m \in \prod_{i=1}^{n+1} M_i,$$

$$m = i_k - i_{k+1} \ \text{ with } \ i_k - i_{k+1} \in \bar{I}_k, \ m \in \bar{I}_k \cap \prod_{i=1}^{n+1} M_i, \ \bar{I}_k \subseteq \bar{I}_{k+1} + \bar{I}_k \cap \prod_{i=1}^{n+1} M_i,$$

$$\bar{I}_k = \bar{I}_{k+1} + \bar{I}_k \cap \prod_{i=1}^{n+1} M_i \ \ \forall k \geq \bar{n}.$$

Hence:

$$\bar{I}_{\bar{n}} = \bar{I}_{\bar{n}+1} + \bar{I}_{\bar{n}} \cap \prod_{i=1}^{n+1} M_i, \ \bar{I}_{\bar{n}+1} = \bar{I}_{\bar{n}+2} + \bar{I}_{\bar{n}+1} \cap \prod_{i=1}^{n+1} M_i, \ \bar{I}_{\bar{n}+2} = \bar{I}_{\bar{n}+3} + \bar{I}_{\bar{n}+2} \cap \prod_{i=1}^{n+1} M_i ...$$

We define a new chain:
$(\bar{I}_{\bar{n}} \cap \prod_{i=1}^{n+1} M_i) \supset (\bar{I}_{\bar{n}+1} \cap \prod_{i=1}^{n+1} M_i) \supset (\bar{I}_{\bar{n}+2} \cap \prod_{i=1}^{n+1} M_i) \supset ....$
Since $\bar{I}_{\bar{n}} \supset \bar{I}_{\bar{n}+1} \supset \bar{I}_{\bar{n}+2}...$ does not become stationary, the new chain does not become stationary, because $\bar{I}_k \cap \prod_{i=1}^{n+1} M_i \neq \bar{I}_{k+1} \cap \prod_{i=1}^{n+1} M_i \ \ \forall k \geq \bar{n}$. Infact, by absurd, if we suppose that there exists $\bar{k} \geq \bar{n}$ such that
$\bar{I}_{\bar{k}} \cap \prod_{i=1}^{n+1} M_i = \bar{I}_{\bar{k}+1} \cap \prod_{i=1}^{n+1} M_i$, then:

$$\bar{I}_{\bar{k}} = \bar{I}_{\bar{k}+1} + \bar{I}_{\bar{k}} \cap \prod_{i=1}^{n+1} M_i = \bar{I}_{\bar{k}+1} + \bar{I}_{\bar{k}+1} \cap \prod_{i=1}^{n+1} M_i = \bar{I}_{\bar{k}+1}, \text{ absurd.}$$

Hence the new chain is a infinite strictly decreasing chain and it is contained in $\prod_{i=1}^{n+1} M_i \subsetneq \prod_{i=1}^{n} M_i$. Repeating the same proof for this new chain, and then possibly for another chain derived from it, and so on, after a finite steps (because the ring is semilocal) we obtain a chain contained in $J(R)$, which does not become stationary. Absurd. □

The following example is a very simple application of Theorem 5.2.

Example 5.3. *If R is a J-semisimple commutative ring with unity, then the following are equivalent:*
(a) R is artinian;
(b) R is semilocal.

Proof.$(a) \Rightarrow (b)$ Trivial.
$(b) \Rightarrow (a)$ If R is J-semisimple, then J(R)=(0) by definition. The result follows from Theorem 5.2. □

6 Conclusions

In recent years several papers have been published on the topics of ideals of computable rings. In 2010 C.J. Conidis proved some results regarding the chain conditions of this kind of rings [3]. Also it is well known that the following is an arithmetical characterization of the Jacobson radical of a computable ring:

$$r \in J(R) \Leftrightarrow (\forall x \in R)(\exists a \in R)[(1 - rx)a = 1].$$

Since $\{(r, x, a)|(1 - rx)a = 1\}$ is computable, it follows that $J(R)$ is $\prod_2^0$, more precisely $\prod_2^0$ −complete by Theorem 4.4 of [4]. Therefore Theorem 5.1 and Theorem 5.2 of the present paper could be useful for algorithmic optimizations and more generally for the computable rings theory.

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