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Feebly J-Clean Unital Rings

Peter V. Danchev

Department of Mathematics
University of Plovdiv, Bulgaria

Abstract

We define the class of feebly J-clean unital rings and establish a characterization theorem giving up to an isomorphism their algebraic structure.

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1. Introduction and Background

Everywhere in the text all our rings are assumed to be associative, containing the identity element 1 which differs from the zero element 0. Our terminology and notations are mainly in agreement with [4]. For instance, $U(R)$ stands for the group of units in R, $J(R)$ for the Jacobson radical of R, $Nil(R)$ for the set of nilpotents in R and $Id(R)$ for the set of idempotents in R.

Before presenting our chief achievements, we will foremost provide a brief history of the principally known things in the subject: It was introduced in [5] the class of *clean* rings R that are rings R whose elements are the sum of a unit and an idempotent. Besides, it was defined in [6] the (smallest) class of *semi-boolean* rings as those rings R for which every element is the sum of an element from the Jacobson radical and an idempotent; these rings were also termed *J-clean*. This was extended in [2] to the class of rings called *weakly semi-boolean* as being rings whose elements are the sum or the difference of an element from the Jacobson radical and an idempotent; these rings were also termed *weakly J-clean*.

On the other vein, in order the enlarge in a non-trivial way the class of clean rings, in [1] were investigated the so-called *feebly clean* rings as rings R, each element $r \in R$ being presentable like this: $r = u + e - f$, where $u \in U(R)$ and $e, f \in Id(R)$ with $ef = fe = 0$.

So, the aim of the present paper is to set and explore in detail the next class of rings as follows:

Definition 1.1. We shall say that a ring R is *feebly J-clean* if, for any $r \in R$, there exist $j \in J(R)$ and $e, f \in Id(R)$ with $ef = fe = 0$ such that $r = j+e-f$.

In addition, if j commutes with (either) e or f, the feebly J-clean ring R having this property is said to be *strongly feebly J-clean*.

Clearly, the following relationships are obviously fulfilled:

$$\{\text{weakly J-clean}\} \subseteq \{\text{feebly J-clean}\} \subseteq \{\text{feebly clean}\}.$$

Observe that if $2 \in J(R)$, then the concepts of feebly J-clean and J-clean do coincide, because $r = j + e - f = (j - 2f) + (e + f) \in J(R) + Id(R)$.

The basic results we obtained the following ones:

2. Main Results

We first begin with a few technicalities:

Lemma 2.1. *If R is a feebly J-clean ring, then $6 \in J(R)$.*

Proof. Writing $2 = j + e - f$ for $j \in J(R)$ and $e, f \in Id(R)$ with $ef = fe = 0$, we have that $2 - j = e - f$ and by squaring it follows that $4 - 4j + j^2 = e + f$ is an idempotent. Consequently, $(4 - 4j + j^2) = 4 - 4j + j^2$, which easily implies that $12 \in J(R)$. Hence $6^2 = 3.12 \in J(R)$ yielding that $6 + J(R)$ is a central nilpotent in $R/J(R)$. Since this quotient ring does not have non-trivial central nilpotents, we deduce that $6 \in J(R)$, as claimed. □

The following generalizes [2, Proposition 3.4].

Proposition 2.2. *Feebly J-clean rings are clean.*

Proof. For an arbitrary element $r \in R$ write $r - 1 = j + e - f$, where $j \in J(R)$ and $e, f \in Id(R)$ with $ef = fe = 0$. But one sees that $r = [j+(1-2f)]+(e+f)$. Since $(1 - 2f)^2 = 1$ and $(e + f)^2 = e + f$, it follows that $r \in (J(R) + U(R)) + Id(R) = U(R) + Id(R)$, as required. □

Recall that a ring R is called *tripotent* if each its element satisfies the equality $x^3 = x$. Moreover, we remember that *idempotents lift modulo* $J(R)$ if for any $a \in R$ with $a^2 - a \in J(R)$ there is $e \in Id(R)$ such that $e - a \in J(R)$. By analogy, we shall say that *tripotents lift modulo* $J(R)$ if, for $b \in R$, whenever $b^3 - b \in J(R)$ there is $t \in R$ with $t^3 = t$ such that $t - b \in J(R)$. More specially, we will say that *tripotents feebly lift modulo* $J(R)$ if, for $c \in R$, the condition $c^3 - c \in J(R)$ leads to the existence of two idempotents $e, f \in R$ with $ef = fe$

such that $(e-f)-c \in J(R)$. Note once again that $e-f$ is always a tripotent element in R and so feebly lifting of tripotents yields the ordinary tripotent's lifting.

We now come to our central structural result, thus expanding [6, Lemma 24] and [2, Theorem 3.5].

Theorem 2.3. *For a ring R the next three items are equivalent:*
(i) R is feebly J-clean;
(ii) $R/J(R)$ is tripotent and tripotents feebly lift modulo $J(R)$;
(iii) $R/J(R) \subseteq \prod_\lambda \mathbb{Z}_2 \times \prod_\mu \mathbb{Z}_3$, where λ, μ are some ordinals, and tripotents feebly lift modulo $J(R)$.

Proof. The equivalence (ii) $\iff$ (iii) is done in [3]. So, we will be concentrating on the equivalence between (i) and (ii). To that goal, assuming (i) holds, we first see with the aid of Definition 1.1 that $R/J(R)$ has to be the sum of two commuting idempotents. We therefore apply [3, Theorem 1] to get that this quotient ring is tripotent, indeed. To show that the feebly lifting is true, for any $r \in R$ we standardly write $r = j + e - f$, where $j \in J(R)$ and $e, f \in Id(R)$ are commuting idempotents. It is routinely checked that the element $e - f$ is always a tripotent, i.e., $(e-f)^3 = e-f$, whence it can be verified at once that the containment $r^3 - r \in J(R)$ is always valid, as expected.

Reciprocally, assume now that (ii) holds. So, taken $r \in R$, it follows that $(r + J(R))^3 = r^3 + J(R) = r + J(R)$, i.e., $r^3 - r \in J(R)$. By supposition there are $e, f \in Id(R)$ which commute such that $e - f - r \in J(R)$. Finally, $r \in e - f + J(R)$, and we are finished. □

In some partial cases we can say a bit more:

Corollary 2.4. *Suppose that R is a ring of characteristic 3. Then R is feebly J-clean if, and only if, $R/J(R)$ is tripotent and tripotents lift modulo $J(R)$ if, and only if, $R/J(R) \subseteq \prod_\mu \mathbb{Z}_3$, where μ is an ordinal, and tripotents lift modulo $J(R)$.*

Proof. In accordance with Theorem 2.3 (ii) and (iii), it suffices to show that the tripotent's lifting does imply the feebly lifting of tripotents. But this follows readily by the presentation $t - 1 = t^2 - (1 - (t - t^2))$, provided $t^3 = t$; notice that $t - 1$ is also a tripotent because $(t-1)^3 = t^3 - 1 = t - 1$ as well as both t^2 and $t - t^2$ are commuting idempotents. Since for all $c \in R$ it must be that $c^3 - c = (c^3 + 1) - (c + 1) = (c + 1)^3 - (c + 1) \in J(R)$ and so $t - (c + 1) = (t - 1) - c \in J(R)$, by what we have just shown the claim follows. □

As an immediate consequence, we derive:

Corollary 2.5. *For any non-zero ring R and any natural $n \geq 2$ the full matrix $n \times n$ ring $\mathbb{M}_n(R)$ is not feebly J-clean.*

Proof. If they were, in view of Theorem 2.3 combined with [4], the factor-ring $\mathbb{M}_n(R)/J(\mathbb{M}_n(R)) = \mathbb{M}_n(R)/\mathbb{M}_n(J(R)) \cong \mathbb{M}_n(R/J(R))$ would be commutative and, even much more, each its element must be the sum of two idempotents. However, by virtue of [3, Lemma 3] this is impossible, thus concluding the assertion. □

The following query arises rather naturally.

Problem 2.6. Describe uniquely feebly J-clean rings, that are feebly J-clean rings for which the existing idempotents are unique.

Are they abelian rings?

We end the work with a technical corrigendum.

Correction. In [2], on p. 267, line 16, there is a misprint, namely the symbol R should be written and read as $R/J(R)$.

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