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Weakly Square-Nil-Fine Rings

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Abstract

We define a proper subclass of the class of fine rings introduced by Călugăreanu-Lam in J. Algebra Appl. (2017) and completely characterize the new class by showing that all rings from it are isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$. This supplies two recent publications by Danchev in International Math. Forum (2016,2017) concerning invo-fine rings.

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1. Introduction and Background

Everywhere in the text of the present article, all our rings R are assumed to be associative, containing the identity element 1, which differs from the zero element 0. Our terminology and notations are mainly in agreement with [9]. For instance, $Nil(R)$ denotes the set of all nilpotent elements in R, and $Nil_2(R)$ is its subset consisting of all nilpotents of order less than or equal to 2. Likewise, $U(R)$ stands for the unit group of R with a subset $Inv(R)$ consisting of all invertible elements of order not exceeding 2. As usual, $J(R)$ is the Jacobson radical of R.

Following [8] and [2], respectively, a ring R is said to be UU provided that $U(R) = 1 + Nil(R)$ and WUU provided that $U(R) = \pm 1 + Nil(R)$.

Imitating [1], a ring R is called *fine* if the equality $R \setminus \{0\} = U(R) + Nil(R)$ holds (two their substantial generalizations to the so-called *nil-good* rings was done in [3] and to the so-called *unit-nil* rings was done in [6]). This is a quite large class of rings and so rather difficult for a complete structural description.

However, it was shown in [1] that fine UU rings are exactly isomorphic to $\mathbb{Z}_2$. Moreover, in [7] we proved that fine WUU rings of index of nilpotence at most 3 are isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Mimicking [4] or [5] (see also [3]), a ring R is called *invo-fine* if the equality $R \setminus \{0\} = Inv(R) + Nil(R)$ holds. These rings form a proper subclass of the class of fine rings and was completely characterized in [5] as being isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ as well.

Of some interest is then to consider rings whose elements are expressed as sums of certain nilpotents. So, we are now in a position to state the following notion.

Definition 1.1. A ring R is said to be *square-nil-fine* if $R\setminus\{0\} = 1+Nil_2(R)+Nil(R)$.

We slightly extend this as follows:

Definition 1.2. A ring R is said to be *weakly square-nil-fine* if $R \setminus \{0\} = \pm 1 + Nil_2(R) + Nil(R)$.

These two sorts of rings arise rather naturally in different aspects of the theory. So, the objective of the current paper is to describe completely the structure of (weakly) square-nil-fine rings up to an isomorphism. Namely, we will show that they are definitely isomorphic to one of the fields $\mathbb{Z}_2$ or $\mathbb{Z}_3$. This will be done in the subsequent section.

2. Main Results

Our starting point is the complete characterization of rings from Definition 1.1. Surprisingly, these rings are of necessity commutative and even much more – they contain only two elements.

Theorem 2.1. *A ring R is square-nil-fine if, and only if, $R \cong \mathbb{Z}_2$.*

Proof. "$\Leftarrow$". We claim $2 = 0$. If not, we can write that $2 = 1 + q + t$, where $q \in Nil(R)$ and $t \in Nil_2(R)$. Hence $1 - q = t \in U(R) \cap Nil_2(R) = \emptyset$, a contradiction. Thus $2 = 0$, indeed. But $t^2 = 0$ forces that $(1 + t)^2 = 1$ whence $R \setminus \{0\} = 1 + Nil_2(R) + Nil(R) \subseteq Inv(R) + Nil(R)$. Therefore, the basic idea from [5] works to get the desired assertion that R is the field of two elements, as promised.

"$\Rightarrow$". It is self-evident. □

We now arrive at the following complete description of rings in Definition 1.2. As above, it is surprising the fact that these rings are necessarily commutative and even much more – they contain two or three elements only.

Theorem 2.2. *A ring R is weakly square-nil-fine if, and only if, $R \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$.*

Proof. "$\Leftarrow$". We assert that either $2 = 0$ or $3 = 0$. If the first is true, then it is fairly clear that R is square nil-fine and, consequently, Theorem 2.1 is applicable to deduce that R is the two element field. If now $2 \neq 0$, then one can write that $2 = 1 + q + t$ or $2 = -1 + q + t$, where q is a nilpotent and t is a square nilpotent. Thus either $1 - q = t \in U(R) \cap Nil_2(R)$ which is impossible, or $3-q = t$. Squaring this, one sees that $3^2 = 9 = 6q-q^2 \in Nil(R)$ that implies $3 \in Nil(R)$ and $2 \in U(R)$. Let us now $3 \neq 0$. So, one writes that $3 \in 1 + Nil(R) + Nil_2(R)$ or $3 \in -1 + Nil(R) + Nil_2(R)$. That is why, $2 + Nil(R) \subseteq U(R) \cap Nil_2(R)$ or $4 + Nil(R) \subseteq U(R) \cap Nil_2(R)$ which containments are both false. Finally, $3 = 0$ must be fulfilled, as expected.

Furthermore, we claim that $Nil_2(R) = Nil(R) = \{0\}$. To this goal, assuming the contrary, given $0 \neq z \in Nil_2(R)$, we write that $z = 1 + q + t$ or $z = -1 + q + t$, where $q \in Nil(R)$ and $t \in Nil_2(R)$. Since there is $n \in \mathbb{N}$ big enough with $q^{3^n} = 0$, we infer that $(z - t)^{3^n} = (1 + q)^{3^n} = 1$ or $(z-t)^{3^n} = (-1+q)^{3^n} = -1$.

We shall deal now with the first equality. It is a routine technical matter to verify that

- $(z - t)^3 = -ztz - tzt$;
- $(z - t)^9 = -(zt)^4 z - (tz)^4 t$;

and etc., by ordinary induction, we derive

- $(z - t)^{3^n} = -(zt)^k z - (tz)^k t$ for some natural number k depending on n.

It follows now that $-(zt)^k z-(tz)^k t = 1$ and thus, multiplying both sides by z and t on different sides, we derive that $zt = tz = 0$. This, in turn, assures that $0 = 1$ that is manifestly not valid. By the same token we can process the other equality $(z-t)^{3^n} = -1$ which amounts to $(t-z)^{3^n} = 1$. And so, finally, $z = 0$ which yields that R contains only the elements $\{-1, 0, 1\}$, as required.

"$\Rightarrow$". It is self-trivial. $\square$

What can be said after all is that weakly square-nil-fine rings are precisely the invo-fine rings, and vice versa. On the other vein, a question which immediately arises is to classify those rings R for which $R = Nil_2(R) + Nil(R)$ or $R = 1 + Nil_2(R) + Nil(R)$, respectively for which $R = Nil_2(R) + Nil(R)$ or $R = \pm 1 + Nil_2(R) + Nil(R)$. Here the ring $\mathbb{Z}_4$ surprisingly occurred, that is not even a fine ring as simple calculations show.

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