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Some Results About Generalized BCH-Algebras

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Abstract

We give a new notion, the notion of a generalized BCH-algebra. This notion of generalized BCH-algebra is so general that the class of generalized BCH-algebras contains the class of BCH-algebras, the class of BCI-algebras and the class of BCK-algebras. We also describe the minimal elements and branches of a generalized BCH-algebra X. The collection of minimal elements is a subalgebra of X and the branches of X are pairwise disjoint and constitute a partition of X. The BCA-part of X is defined and is investigated. Medial generalized BCH-algebras are defined and their relation with BCI-algebras is investigated.

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1. Introduction

In 1966, two classes of abstract algebras, BCK-algebras and BCI-algebras, were introduced by Y. Imai and K. Iséki [11, 12]. The notion of a BCI-algebra is a generalization of the notion of a BCK-algebra. It is known that BCK and BCI algebras are inspired by some implicational logic [12]. These algebras have been studied extensively by various researchers during the past fifty years (see [8], [15], [14], [13], [18], [20], [21] and references therein).

Logical systems are important ingredients of artificial intelligence, used for human decision making. Along with classical logical systems, some logical systems dealing with uncertainty of information have been developed using many-valued logic as well as fuzzy logic. The study of BCK-algebras and BCI-algebras have been inspired by a BCK positive logic and BCI positive logic, respectively. The link between these algebras and their corresponding logics is strong to the extent that translation procedures have been developed for formulas and theorems of a logic in terms of the corresponding algebras. Thus the study of abstract algebras, motivated by known logics, and their generalizations is of great interest for the researchers working in the fields of algebraic structures, artificial intelligence and computing. Thus generalizations of BCK/BCI-algebras have been investigated by various researchers (see [2], [3], [4], [6], [7], [9], [10], [16], [17] and references therein).

In 1985, Hu and Li [9, 10] introduced the notion of a BCH-algebra, which is a generalization of the notions of BCK and BCI-algebras. They have studied a few properties of these algebras. Some other properties of these algebras have been studied by Chaudhry [3, 4] Dudek and Thomys [7] and many other researchers (See [1], [19] and references therein).

Keeping in view the importance of above mentioned algebras as algebraic structures as well as in the field of artificial intelligence and computing, we introduce a new class of abstract algebras, called generalized BCH-algebras. This class of generalized BCH-algebras is so general that it contains the class of BCH-algebras, the class of BCI-algebras and the class of BCK-algebras. Thus generalized BCH-algebras have strong connections with BCK positive logic, BCI positive logic and the systems, based upon these logics, used in decision making processes by the researchers in artificial intelligence and computer science. We feel that generalized BCH-algebras may be of great help in formulating logical systems which will be more general than the logical systems based upon above mentioned logics.

The purpose of this paper is to investigate some properties of generalized BCH-algebras. We define minimal elements and branches of a generalized BCH-algebra. We also show that the collection of all minimal elements of a generalized BCH-algebra X form a subalgebra of X. Further, we show that the branches of a generalized BCH-algebra X are pairwise disjoint and the union of all branches is the whole algebra X. We define BCA-part of a generalized BCH-algebra and study its properties. We also define a medial generalized BCH-algebra and investigate conditions under which a generalized BCH-algebra is a medial BCI-algebra.

2. Preliminaries

In this section we give those definitions and known results about BCK-algebras, BCI-algebras and BCH-algebras which are relevant to the results contained in the next section.

Definition 1. [11, 14] A BCK-algebra $X = (X, *, 0)$ is a non-empty set X with a binary operation$*$ and a distinguished element $0 \in X$ satisfying the following conditions for all $x, y, z \in X$:

(1) $((x * y) * (x * z)) * (z * y) = 0$,
(2) $(x * (x * y)) * y = 0$,
(3) $x * x = 0$,
(4) $0 * x = 0$,
(5) $x * y = 0$ and $y * x = 0$ imply $x = y$.

In 1966, K. Iséki introduced another notion, called BCI-algebra [12], which is a generalization of the notion of a BCK-algebra.

Definition 2. [13] A BCI-algebra $X = (X, *, 0)$ is a non-empty set X with a binary operation $*$ and a particular element $0 \in X$ satisfying the condition $(1), (2), (3), (5)$ given in Definition 1 and

(6) $x * 0 = 0$ implies $x = 0$, $x \in X$.

It is known that every BCK-algebra is a BCI-algebra but the converse is not true [13]. Thus the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Many researchers have investigated different aspects of these algebras (see [8], [9], [10], [15], [14], [18], [21] and reference therein).

In 1985, Q. P. Hu and X. Li introduced a wider class of abstract algebras, called BCH-algebras ([9] and [10]). They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Various researchers ([3], [4], [7], [19] and the references therein) have investigated some properties of this general class of abstract algebras.

Definition 3 ([11, 15])**.** A BCH-algebra $X = (X, *, 0)$ is a non-empty set X with a binary operation $*$ and a distinguished element $0 \in X$ satisfying the following conditions for all $x, y, z \in X$:

(3) $x * x = 0$,
(5) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(7) $(x * y) * z = (x * z) * y$.

Definition 4 ([3])**.** A BCH-algebra is called a proper BCH-algebra if it is not a BCI-algebra.

Example 2.1 ([3])**.** *Let* $X = \{0, 1, 2, 3, 4\}$ *with a binary operation* $*$ *defined by:*

*	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Routine calculation give that this algebra is a BCH-algebra but it is not a BCI-algebra because:
$(((1 * 3) * (1 * 2)) * (2 * 3)) = (1 * 0) * 0 = 1 \neq 0$.

Definition 5 ([9])**.** Let $(X, *, 0)$ be a BCH-algebra. We define the relation $\leq$ on X by $x \leq y$ if and only if $x * y = 0$.

It is known that every BCK/BCI-algebra is a partially ordered set with respect to this relation $\leq$ ([14]). Further, this relation is transitive in BCK-BCI-algebras [14], but this relation is only reflexive and is not transitive, in general, in case of BCH-algebras ([3]).
In [9] and [10], it has been shown that the following results hold in a BCH-algebra.

Theorem 2.2. *Let* $(X, *, 0)$ *be a BCH-algebra. Then the following hold for all* $x, y, z \in X$*:*

(2) $(x * (x * y)) * y = 0$,
(6) $x * 0 = 0$ *implies* $x = 0$,
(8) $x * 0 = x$,
(9) $x \leq y$ *implies* $x * z \leq y * z$,
(10) $x \leq y$ *implies* $z * y \leq z * x$.

Since every BCK/BCI-algebra is a BCH-algebra, so the relations (2), (6), (8), (9) and (10) valid for BCH-algebras are also valid for BCK/BCI-algebras.

Theorem 2.3 ([10])**.** *A BCH-algebra* X *is a proper BCH-algebra if*

(1) $((x * y) * (x * z)) * (z * y) = 0$, $x, y, z \in X$,

does not hold in X*.*

For a BCH-algebra X, some special subsets described below, play an important role in the investigation of its properties.

Definition 6. Let $(X, *, 0)$ be a BCH-algebra. Then a non-empty subset Y of X is called a subalgebra of X if for $x, y \in Y$, $x * y \in Y$.

Definition 7. Let $(X, *, 0)$ be a BCH-algebra (BCI-algebra), then the subset

$$M = \{x : x \in X \, and \, 0 * x = 0\}$$

is called the BCA-part (BCK-part) of X.

It is known that in case of a BCI-algebra, its BCK-part is a BCK-algebra [14], where as the following example shows that in case of a BCH-algebra X, its BCA-part is not a BCI-algebra, in general.

Example 2.4. *Let $X = \{0, 1, 2, 3, 4\}$ be the BCH-algebra of Example 2.1. Then its BCA-part is $M = \{0, 1, 2, 3\}$. Since $(1 * 3) * (1 * 2) = 1 * 0 = 1 \nleq 2 * 3 = 0$, so its BCA-part is not a BCI-algebra.*

Remark 2.5. *We know that in a BCI-algebra the relation $\leq$ is transitive and $x \leq y$ implies $x * y \leq y * z$ and $z * x \leq z * y$. But from above example, we conclude that these relations are not true in case of a BCH-algebra. This is because $1 \leq 2$ and $2 \leq 3$ but $1 \nleq 3$. Further $2 \leq 3$ but $1 * 3 = 1 \nleq 2 * 3 = 0$.*

Definition 8. Let $X, *, 0)$ be a BCH-algebra. A non-empty subset I of X is called an ideal of X if:

(11) $0 \in I$.
(12) $x * y \in I$, $x \in I$ imply that $y \in I$.

Remark 2.6. *It is known that an ideal I of a BCH-algebra X is not a subalgebra, in general* [3]*. To overcome this difficulty the notion of a closed ideal has been defined in* [3]*.*

Definition 9 (Closed Ideal)**.** A non-empty subset B of a BCH-algebra X is called a closed in X or a closed ideal of X if:

(11) $0 * x \in B$. for all $x \in B$,
(12) $y * x \in B$, $x \in B$ imply that $y \in B$.

Remark 2.7. *Since B is non-empty so there is an element $x \in B$. Thus* (11) *gives $0 * x \in B$, so $0 * x \in B$, $x \in B$. By* (12) *we get $0 \in B$. Let $x, y \in B$ then $(x * y) * x = 0 * y \in B$. By* (12) *we get $x * y \in B$. Thus every closed ideal is a subalgebra but converse is not true, in general.*

We consider the BCH-algebra of Example 2.1 and take $B = \{0, 4\}$. Then B is a subalgebra of X but is not an ideal because $4 = 1 * 4 \in B$, $4 \in B$ but $1 \notin B$.

3. Main Results

We now describe our notion of a generalized BCH-algebra and give our results about these algebras.

4. Generalized BCH-algebras

Definition 10 (Generalized BCH-algebra)**.** A non-empty set X together with a fixed element 0 and a binary operation $*$ is called a generalized BCH-algebra if it satisfies the following conditions:

(1) $(GBCH-1) : x * x = 0$,
(2) $(GBCH-2)$: If $x * y = 0$ and $y * x = 0$ then $x = y$,
(3) $(GBCH-3) : (x * (x * y)) * y = 0$,
(4) $(GBCH-4) : (x * y) * x = (x * x) * y$,

for all $x, y \in X$.

Remark 4.1. *Every BCH-algebra X is a generalized BCH-algebra. This is because $(x * y) * z = (x * z) * y$ for al $x, y, z \in X$, implies $(x * (x * y)) * y = (x * y) * (x * y) = 0$, $(x * y) * x = (x * x) * y$, by replacing z by $x * y$ and z by x, respectively, in the condition $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.*

The following example shows that the converse of Remark 4.1 is not true, in general.

Example 4.2. *Let $X = \{0, 1, 2, a, b\}$ with binary operation $*$ defined by:*

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	0	a	a
2	2	2	0	a	b
a	a	a	a	0	0
b	b	a	a	1	0

Then routine calculations show that X is a generalized BCH-algebra but X is not a BCH-algebra because $(2 * a) * b = a * b = 0$ and $(2 * b) * a = b * a = 1$, implies $(2 * a) * b \neq (2 * b) * a$.

Definition 11 (Proper generalized BCH-algebra)**.** A generalized BCH-algebra $(X, *, 0)$ is called proper if it is not a BCH-algebra.

Remark 4.3. *Example 4.2 shows that proper generalized BCH-algebra exist. Moreover, Remark 4.1 implies that the class of generalized BCH-algebras contains the class of BCH-algebras and hence the class of BCI-algebras as well as the class of BCK-algebras.*

We now give another example of a proper generalized BCH-algebra.

Example 4.4. *Let $X = \{0, a, b, c, d, e, f, g, h, i, j, k, l, m, n, n, p, q, r, s, t, u, v, w, x, y\}$ with binary operation $*$ defined, on next page, by:*

*	0	a	b	c	d	e	f	g	h	i	j	k	l	m	n	p	q	r	s	t	u	v	w	x	y
0	0	0	0	0	0	0	0	0	0	i	i	i	i	i	i	p	p	p	p	p	p	v	v	v	v
a	a	0	0	0	0	a	a	0	0	i	i	i	i	i	i	q	p	p	q	p	p	v	v	v	v
b	b	e	0	0	e	a	a	0	0	l	l	i	i	i	i	q	p	p	q	p	p	v	v	v	v
c	c	c	d	0	e	d	d	d	0	l	k	j	i	i	j	r	r	p	r	r	p	v	w	v	w
d	d	d	d	0	0	d	d	d	0	i	j	j	i	i	j	r	r	p	r	r	p	v	w	v	w
e	e	e	0	0	e	0	0	0	0	l	l	i	i	i	i	p	p	p	P	p	p	v	v	v	v
f	f	f	f	f	f	f	0	0	0	m	m	m	m	i	i	P	p	p	s	s	s	v	v	x	x
g	g	f	f	f	f	g	a	0	0	m	m	m	m	i	i	q	p	p	t	s	s	v	v	x	x
h	h	h	h	f	f	h	d	d	0	m	n	n	m	i	j	r	r	p	u	u	s	v	w	x	y
i	i	i	i	i	i	i	i	i	i	0	0	0	0	0	0	v	v	v	v	v	v	p	p	p	p
j	j	i	i	i	i	j	j	i	i	a	0	0	a	a	0	w	v	v	w	v	v	q	p	q	p
k	k	l	i	i	l	j	j	i	i	b	e	0	a	a	0	w	v	v	w	v	v	q	p	q	p
l	l	l	i	i	l	i	i	i	i	e	e	0	0	0	0	v	v	v	v	v	v	p	p	p	p
m	m	m	m	m	m	m	i	i	i	f	f	f	f	0	0	v	v	v	x	x	x	p	p	s	s
n	n	m	m	m	m	n	j	i	i	g	f	f	g	a	0	w	v	v	y	x	x	q	p	t	s
p	p	p	p	p	p	p	p	p	p	v	v	v	v	v	v	0	0	0	0	0	0	i	i	i	i
q	q	p	p	p	p	q	q	p	p	v	v	v	v	v	v	a	0	0	a	0	0	i	i	i	i
r	r	r	r	p	p	r	r	r	p	v	w	w	v	v	w	d	d	0	d	d	0	i	j	i	j
s	s	s	p	p	s	p	p	p	p	x	x	v	v	v	v	e	e	e	0	0	0	l	l	i	i
t	t	s	p	p	s	q	q	p	p	x	x	v	v	v	v	b	e	e	a	0	0	l	l	i	i
u	u	u	r	p	s	r	r	r	p	x	y	w	v	v	w	c	c	e	d	d	0	l	k	i	j
v	v	v	v	v	v	v	v	v	v	p	p	p	p	p	p	i	i	i	i	i	i	0	0	0	0
w	w	v	v	v	v	w	w	v	v	q	p	p	q	q	p	j	i	i	j	i	i	a	0	a	0
x	x	x	v	v	x	v	v	v	v	s	s	p	p	p	p	l	l	l	i	i	i	e	e	0	0
y	y	x	v	v	x	w	w	v	v	t	s	p	q	q	p	k	l	l	j	i	i	b	e	a	0

Routine calculations give that X is a generalized BCH-algebra. We note that $(g*r)*y = p*y = i$ and $(g*y)*r = x*r = l$. Hence $(g*r)*y \neq (g*y)*r$. Thus X is not a BCH-algebra. Hence X is a proper generalized BCH-algebra. We also note that $(h*y)*(h*v) = y*v = b \neq 0 = v*y$. Thus $(h*y)*(h*v) \nleq v*y$, whereas it is known [3] that a BCH-algebra X satisfies the relation $(h*y)*(h*v) \leq v*y$ for all $h, v, y \in X$.

In the sequel we shall adopt the same definitions of the notions, the relation $\leq$, subalgebra, BCA-part and the closed ideals for generalized BCH-algebras as are given for these notions regarding BCH/BCI-algebras in the previous section.

Definition 12 (Minimal element)**.** An element $x_0 \in X$ is called a minimal element of X if $x \leq x_0$ implies $x = x_0$.

The collection of all minimal elements of a generalized BCH-algebra X is denoted by $Min(X)$.

Example 4.5. (*a*) *Let $(X, *, 0)$ be a generalized BCH-algebra of Example 4.2. Then 0 and a are minimal elements of X.*

(*b*) *Let $(X, *, 0)$ be a generalized BCH-algebra of Example 4.4. Then 0, p, i and v are minimal elements of X.*

Definition 13 ($Med(X)$, Medial element of X)**.** Let $(X, *, 0)$ be a generalized BCH-algebra. We define:

$$Med(X) = \{x : x \in X \text{ and } 0*(0*x) = x\} \subseteq X.$$

The set $Med(X)$ is called the medial part of X.

An element $x \in Med(X)$ is called a Medial element of X. Thus for a medial element x of X, $0*(0*x) = x$.

Proposition 4.6. *In a generalized BCH-algebra $(X, *, 0)$ the following hold:*

(1) *0 is a minimal element of X,*

(2) *$x*0 = x$ for all $x \in X$.*

Proof. (1). Let $x \in X$ such that $x \leq 0$. Thus $x*0 = 0$. Now $0*x = (x*0)*x = (x*(x*x))*x = 0$. Thus $x*0 = 0*x = 0$. Hence $x = 0$. So 0 is a minimal element of X.

(2). Obviously $x*(x*0) \leq 0$. Since 0 is a minimal element of X, so $x*(x*0) = 0$. Thus $x \leq x*0$. Moreover $(x*0)*x = (x*(x*x))*x = 0$. Hence $x*0 \leq x$. Thus $x*0 = x$. □

Remark 4.7. *Since in a generalized BCH-algebra X, $0*(0*0) = 0$, so $0 \in Med(X)$. From Proposition 4.6 we get that $0 \in Min(X)$. Hence $Med(X)$ and $Min(X)$ are non-empty.*

Proposition 4.8. *Let $(X, *, 0)$ be a generalized BCH-algebra. Then an element $x_0 \in X$ is a medial element if and only if it is a minimal element.*

Proof. Let x_0 be a medial element of X. Then $0*(0*x_0)=x_0$. We now show that x_0 is a minimal element of X. Let $x \leq x_0$. Then $x*x_0=0$. This implies $(x*x_0)*x=0*x$. That is, $(x*x_0)*x=0*x$. Thus $0*x_0=0*x$, which implies $0*(0*x_0)=0*(0*x)$. Thus $x_0=0*(0*x)\leq x$. Hence $x=x_0$. So x_0 is a minimal element of X.

Conversely let x_0 be a minimal element of X. Since $0*(0*x_0)\leq x_0$ and x_0 is a minimal element of X, so $x_0=0*(0*x_0)$. Thus x_0 is a medial element of X. □

Remark 4.9. *From Proposition 4.8 we get that for a generalized BCH-algebra X, $Min(X)=Med(X)$.*

Theorem 4.10. *Let $(X,*,0)$ be a generalized BCH-algebra. Let $x\in X$, then $0*x\in Min(X)$.*

Proof. Let $x,y\in X$. Let $y\leq 0*x$. Thus $y*(0*x)=0$, which implies $(y*(0*x))*y=0*y$. That is, $(y*y)*(0*x)=0*y$. Thus $0*(0*x)=0*y$. Now $(0*y)*x=(0*(0*x))*x=0$. Thus $((0*y)*x)*(0*y)=(0*(0*y))$. This gives $((0*y)*(0*y))*x=0*(0*y)$. That is, $0*x=0*(0*y)\leq y$. Hence $y=0*x$. Thus $0*x\in Min(X)$.

□

Remark 4.11. *Let $(X,*,0)$ be a generalized BCH-algebra. Let $x\in X$. Then $0*x\in Min(X)$. Since $Min(X)=Med(X)$, therefore $0*x\in Med(X)$. Thus*

$$0*(0*(0*x))=0*x,$$

for all $x\in X$.

Theorem 4.12. *Let $(X,*,0)$ be a generalized BCH-algebra. Then $Min(X)=\{0*x:\, x\in X\}$.*

Proof. Let $B=\{0*x:\, x\in X\}$ and let $y\in B$. Then, $y=0*x$ for some $x\in X$. By Theorem 4.10 $y=0*x\in Min(X)$. So $B\subseteq Min(X)$. Let $x\in Min(X)$. Since $Min(X)=Med(X)$, so $x\in Med(X)$. Thus $x=0*(0*x)=0*y$, where $y=0*x$. Hence $x\in B$. Thus $Min(X)\subseteq B$. So $Min(X)=B=\{0*x:x\in X\}$. □

Remark 4.13. *From above theorem and Remark 4.9, we conclude that if $(X,*,0)$ is a generalized BCH-algebra, then $Min(X)=Med(X)=\{0*x:\, x\in X\}$.*

Definition 14 (Branch of a generalized BCH-algebra)**.** Let x_0 be a minimal element of a generalized BCH-algebra X. Then the set:

$$B(x_0)=\{x:x\in X\, and\, x_0\leq x\}$$

is called the branch of X, determined by x_0.

Example 4.14. (*a*) *Let $(X,*,0)$ be the generalized BCH-algebra of Example 4.2, then $B(0)=\{0,1,2\}$ and $B(a)=\{a,b\}$.*

(b) *Let $(X, *, 0)$ be the generalized BCH-algebra of Example 4.4. Then $B(0) = \{0, a, b, c, d, e, f, g, h\}$, $B(i) = \{i, j, k, l, m, n\}$, $B(p) = \{p, q, r, s, t, u\}$ and $B(v) = \{v, w, x, y\}$*

Theorem 4.15. *Let $(X, *, 0)$ be a generalized BCH-algebra. Then for each $x \in X$, there is a unique $x_0 \in Min(X)$ such that $x_0 \leq x$, that is, x belongs to the unique branch $B(x_0)$, determined by x_0.*

Proof. Let $x \in X$. Then $0 * (0 * x) \leq x$. We take $x_0 = 0 * (0 * x)$. So $x_0 \leq x$. Further by Theorem 4.12, $x_0 = 0 * (0 * x) \in Min(X)$. Thus there is an $x_0 \in Min(X)$ such that $x \in B(x_0)$.

Let $y_0 \in Min(X)$ be such that $x \in B(y_0)$. Thus $y_0 \leq x$, so $y_0 * x = 0$, which gives $(y_0 * x) * y_0 = 0 * y_0$. That is, $0 * x = 0 * y_0$. Thus $0 * (0 * x) = 0 * (0 * y_0)$. So $x_0 = 0 * (0 * y_0) \leq y_0$. Since $y_0 \in Min(X)$, so $x_0 = y_0$. Hence for $x \in X$, there is a unique $x_0 = 0 * (0 * x)$ such that $x \in B(x_0)$. □

Remark 4.16. (a) *From above theorem we conclude that in a generalized BCH-algebra $(X, *, 0)$, for any $x \in X$, there exists a unique $x_0 = 0 * (0 * x) \in Min(X) = Med(X)$ such that $x_0 \leq x$. That is, every $x \in X$ belongs to a unique branch $B(x_0)$, where $x_0 = 0 * (0 * x) \in Min(X) = Med(X)$.*

(b) *The unique $x_0 = 0 * (0 * x)$ for $x \in X$ is called the corresponding minimal element of x or medial element of x.*

Theorem 4.17. *Let $(X, *, 0)$ be a generalized BCH-algebra. Let $x_0, y_0 \in Min(X)$. Then $x_0 = y_0$ if and only if $B(x_0) = B(y_0)$.*

Proof. Let $x_0 = y_0$ and let $B(x_0) \neq B(y_0)$. Without loss of generality we suppose that there is an $x' \in B(x_0)$ such that $x' \notin B(y_0)$. Since $x' \in B(x_0)$ so $x_0 \leq x'$. Thus $x_0 * x' = 0$. This implies $(x_0 * x') * x_0 = 0 * x_0$. Thus $0 * x' = 0 * x_0$, which implies $0 * (0 * x') = 0 * (0 * x_0)$. Since $x_0 \in Min(X) = Med(X)$, so $x_0 = 0 * (0 * x_0)$. Thus $x_0 = 0 * (0 * x')$. Since $x_0 = y_0$, so $y_0 = 0 * (0 * x') \leq x'$. Hence $x' \in B(y_0)$, a contradiction. Thus $B(x_0) = B(y_0)$.

Conversely let $B(x_0) = B(y_0)$. Since $x_0 \in B(x_0)$, so $x_0 \in B(y_0)$. Thus $y_0 \leq x_0$. Similarly $y_0 \in B(y_0) = B(x_0)$ gives $x_0 \leq y_0$. Thus $x_0 = y_0$. □

Theorem 4.18. *Let $(X, *, 0)$ be a generalized BCH-algebra. Let $x_0 \neq y_0$ and $x_0, y_0 \in Min(X)$. Then $B(x_0) \cap B(y_0) = \phi$.*

Proof. Let $x_0 \neq y_0$ and $B(x_0) \cap B(y_0) \neq \phi$. Let $x \in B(x_0) \cap B(y_0)$. Then $x \in B(x_0)$ and $x \in B(y_0)$. By Theorem 4.15, we get $B(x_0) = B(y_0)$, which along with Theorem 4.17 implies $x_0 = y_0$, a contradiction. Hence $B(x_0) \cap B(y_0) = \phi$. □

From theorems 4.15 and 4.18 we get the following corollary.

Corollary 4.19. *Let $(X, *, 0)$ be a generalized BCH-algebra. Let $Min(X)$ be the set of all minimal points of X. Let $B_r(X) = \{B(x_0) : x_0 \in Min(X)\}$ be the collection of all branches of X determined by elements of $Min(X)$. Then*

(*i*) $X = \cup_{x_0 \in Min(X)} B(x_0)$,
(*ii*) $B(x_0) \cap B(y_0) = \phi$, $x_0, y_0 \in Min(X)$ *and* $x_0 \neq y_0$,

that is, $B_r(X)$ *is a partition of* X*.*

Theorem 4.20. *Let* $(X, *, 0)$ *be a generalized BCH-algebra. Then* $0*(x*y) = (0*x)*(0*y)$ *for all* $x, y \in X$*.*

Proof. Since X be a generalized BCH-algebra, so $(x*y)*x = (x*x)*y$ for all $x, y \in X$. Now

$$\begin{aligned}(0*x)*(0*y) &= (((x*y)*(x*y))*x)*(0*y)\\ &= (((x*y)*x)*(x*y))*(0*y)\\ &= (((x*x)*y)*(x*y))*(0*y)\\ &= ((0*y)*(x*y))*(0*y)\\ &= ((0*y)*(0*y))*(x*y) = 0*(x*y).\end{aligned}$$

□

Theorem 4.21. *Let* $(X, *, 0)$ *be a generalized BCH-algebra. Then* $Med(X)$ *is a subalgebra of* X*.*

Proof. Let X be a generalized BCH-algebra. Then $0 = (0*0)*0$, so $0 \in Med(X)$. Let $x, y \in Med(X)$. Then $0*(0*x) = x$ and $0*(0*y) = 0$. Then by above theorem,

$$0*(0*(x*y)) = 0*((0*x)*(0*y)) = (0*(0*x))*(0*(0*y)) = x*y.$$

Hence $x*y \in Med(X)$. So $Med(X)$ is a subalgebra of X. □

Remark 4.22. *If* X *is a generalized BCH-algebra, then* $Med(X) = Min(X)$*. Hence from above theorem we get* $Min(X)$ *is a subalgebra of* X*.*

Remark 4.23. (*a*) *Let* $(X, *, 0)$ *be the generalized BCH-algebra of Example 4.2 then* $Med(X) = \{0, a\}$ *is its subalgebra. But* $Med(X) = \{0, a\}$ *is not an ideal because* $1*a = a \in Med(X)$, $a \in Med(X)$ *but* $1 \notin Med(X)$*. Thus for a generalized BCH-algebra* X, $Med(X)$ *is not an ideal, in general.*

(*b*) *Moreover, the relation* $\leq$ *defined by* $x \leq y$ *if and only if* $x*y = 0$ *for a generalized BCH-algebra is not transitive because it is not transitive for a BCH-algebra* [3]*.*

(*c*) *If* $x \leq y$*, then* $x*z \leq y*z$ *and* $z*y \leq z*x$ *do not hold for BCH-algebras* [3]*. Hence these do not hold for generalized BCH-algebras.*

Theorem 4.24. *Let* $(X, *, 0)$ *be a generalized BCH-algebra. Let* M *be its BCA-part. Then* M *is a closed ideal of* X *and hence a subalgebra of* X*. Further if* $x \in M$, $y \in X - M$*, then* $x*y \in X - M$ *and* $y*x \in X - M$*.*

Proof. Let $M = \{x : x \in X \, and \, 0 * x = 0\}$ be the BCA-part of X. Since $0 * 0 = 0$, so $0 \in M$. Let $x \in M$ and $0 \in M$. Hence $0 * x = 0$. This implies $0 * (0 * x) = 0$. So $0 * x \in M$.

Let $y * x \in M$, $x \in M$. So $0 * (y * x) = 0$ and $0 * x = 0$. Now $0 * (y * x) = (0 * y) * (0 * x) = (0 * y) * 0 = 0 * y$. Thus $0 * y = 0 * (y * x) = 0$. Hence $y \in M$. Thus M is a closed ideal of X.

Further let $x \in M$ and $y \in X - M$. Let $x * y \in M$. Since M is a closed ideal of X and hence is a subalgebra of X, so $(x * y) * x = (x * x) * y = 0 * y \in M$. Thus $0 = 0 * (0 * y) \leq y$. So $0 * y = 0$, which implies $y \in M$, a contradiction. Hence $x * y \in X - M$.

Let $x \in M$, $y \in X - M$ and $y * x \in M$. Since M is a closed ideal so $y * x \in M$, $x \in M$ imply that $y \in M$, a contradiction. Thus $y * x \in M$. □

Remark 4.25. *Let $(X, *, 0)$ be a generalized BCH-algebra. Then its BCA-part is $M = \{x : x \in X \, and \, 0 * x = 0\} = \{x : x \in X \, and \, 0 \leq x\} = B(0)$. Thus $M = B(0)$.*

Since for a generalized BCH-algebra $(X, *, 0)$, $Min(X) = Med(X)$, so in the sequel we shall use $Min(X)$ and $Med(X)$ interchangeably.

Proposition 4.26. *Let $(X, *, 0)$ be a generalized BCH-algebra. Let $x_0, y_0 \in Med(X)$. If $x \in B(x_0)$, $z \in B(y_0)$, then $x * z \in B(x_0 * y_0)$.*

Proof. Let $x \in B(x_0)$, $z \in B(y_0)$. Then $x_0 \leq x$ and $y_0 \leq z$. Thus $x_0 * x = 0$ and $y_0 * z = 0$. Hence $(x_0 * x) * x_0 = 0 * x_0$ and $(y_0 * z) * y_0 = 0 * y_0$. Thus $(x_0 * x_0) * x = 0 * x_0$ and $(y_0 * y_0) * z = 0 * y_0$. So $0 * x = 0 * x_0$ and $0 * z = 0 * y_0$, which imply $0 * (0 * x_0) = 0 * (0 * x)$ and $0 * (0 * y_0) = 0 * (0 * z)$. That is, $x_0 = 0 * (0 * x)$ and $y_0 = 0 * (0 * z)$. Since $Med(X)$ is a subalgebra, so $x_0 * y_0 \in Med(X)$. Also

$$x_0 * y_0 = (0 * (0 * x)) * (0 * (0 * z)) = 0 * ((0 * x) * (0 * z)) = 0 * (0 * (x * z)) \leq x * z.$$

Hence $x * z \in B(x_0 * y_0)$. □

Corollary 4.27. *Let $(X, *, 0)$ be a generalized BCH-algebra. Then $x, y \in B(x_0)$ if and only if $x * y \in B(0)$.*

Proof. Let $x, y \in B(x_0)$, $x_0 \in Med(X)$. Then by previous proposition $x * y \in B(x_0 * x_0) = B(0)$.
Conversely let $x * y \in B(0)$. Then $0 \leq x * y$. Thus $0 * (x * y) = 0$. Hence $0 * (0 * (x * y)) = 0$. So $0 * ((0 * x) * (0 * y)) = 0$, which gives $(0 * (0 * x)) * (0 * (0 * y)) = 0$. So $0 * (0 * x) \leq 0 * (0 * y)$. Since $0 * (0 * y) \in Min(X)$, so $0 * (0 * x) = 0 * (0 * y)$. Thus $x_0 = y_0$. Hence $x, y \in B(x_0)$. □

Theorem 4.28. *Let $(X, *, 0)$ be a generalized BCH-algebra. Let $y_0 \in Med(X)$. If $y \in B(y_0)$ and $x \leq y$, $y \leq z$, then $x, z \in B(y_0)$.*

Proof. Since $y \in B(y_0)$, so $y_0 \leq y$. Thus $y_0 * y = 0$, which gives $(y_0 * y) * y_0 = 0 * y_0$. That is, $(y_0 * y_0) * y = 0 * y_0$. Thus $0 * y = 0 * y_0$. Hence $0 * (0 * y) = 0 * (0 * y_0)$. Since $y_0 \in Med(X)$, so $y_0 = 0 * (0 * y)$.

Now $x \leq y$, so $x*y = 0$ thus $(x*y)*x = 0*x$, which gives $(x*x)*y = 0*x$. So $0*y = 0*x$. Thus $0*(0*y) = 0*(0*x)$. So $0*(0*x) = y_0$. Thus $y_0 = 0*(0*x) \leq x$. So $x \in B(y_0)$.

Now $y \leq z$, so $y*z = 0$. Thus $(y*z)*y = 0*y$, which gives $(y*y)*z = 0*y$. Hence $0*z = 0*y$. Thus $0*(0*y) = 0*(0*z)$. That is, $y_0 = 0*(0*z) \leq z$. So $z \in B(y_0)$. □

5. The relation between generalized BCH-algebras, BCH-algebras, BCI-algebras and Medial BCI-algebras

It is known that every BCK-algebra is a BCI-algebra and every BCI-algebrais is a BCH-algebra. Further we have shown that every BCH-algebra is a generalized BCH-algebra. On the other hand, converse of none of the above three statements is true. Thus the class of generalized BCH-algebra properly contains the classes of BCK-algebras, BCI-algebras and BCH-algebras.

Now we investigate the conditions under which the class of generalized BCH-algebras and the class of BCH-algebras coincide.

Proposition 5.1. *A generalized BCH-algebra* $(X,*,0)$ *is a BCH-algebra if* X *satisfies*

(7) $(x*y)*z = (x*z)*y$,

for all $x, y, z \in X$.

Proof. Since X is a generalized BCH-algebra, so it satisfies $GBCH-1$ and $GBCH-2$ given in Definition 10 of a generalized BCH-algebra. Thus it satisfies conditions (3) and (5) given in the Definition 3 of a BCH-algebra. Further by given hypothesis, it satisfies (7). Hence X is a BCH-algebra.

□

We now state the following result:

Theorem 5.2. [9] *A BCH-algebra* $(X,*,0)$ *is a BCI-algebra if it satisfies:*

(1) $[(x*y)*(x*z)]*(z*y) = 0$ *for all* $x, y, z \in X$.

Combining the Proposition 5.1 and Theorem 5.2 we get the following result.

Theorem 5.3. *A generalized BCH-algebra* $(X,*,0)$ *is a BCI-algebra if it satisfies:*

(1) $((x*y)*(x*z))*(z*y) = 0$ *for all* $x, y, z \in X$,

(1′) $(x*y)*z = (x*z)*y$ *for all* $x, y, z \in X$.

A BCI-algebra $(X,*,0)$ is called a medial BCI-algebra if it satisfies:

(8) $x*(x*y) = y$ for all $x, y \in X$ [3].

We adopt the same definition for a medial generalized BCH-algebras.

Definition 15 (Medial generalized BCH-algebra)**.** A generalized BCH-algebra $(X,*,0)$ is called medial if it satisfies $x*(x*y) = y$ for all $x, y \in X$.

It has been shown in [3] that for a BCI-algebra X, the following are equivalent:

(8) X is medial,

(9) $0 * (0 * x) = x$,

(10) $M = \{0\}$, where M is the BCK-part of X.

The following theorem shows that (8), (9) and (10) are also equivalent for a generalized BCH-algebra.

Theorem 5.4. *Let $(X, *, 0)$ be a generalized BCH-algebra. Then the following are equivalent in X:*

(8) *X is medial,*

(9) $0 * (0 * x) = x$,

(10) $M = \{0\}$, *where M is the BCA-part of X.*

Proof. (8) $\Rightarrow$ (9)
Let X be medial. Then $x * (x * y) = y$ for all $x, y \in X$. Replacing x by 0 in this identity, we get (9).
(9) $\Rightarrow$ (10)
Let $0 * (0 * x) = x$ for all $x \in X$. Let $y \in M$. Then $0 = 0 * y$. This implies $0 * 0 = 0 * (0 * y)$. Thus $0 = y$. Hence $M = \{0\}$.
(10) $\Rightarrow$ (8)
Let $M = \{0\}$ and let $x, y \in X$. Then $x * (x * y) \leq y$ for all $x, y \in X$. Thus $y * (x * (x * y)) \in B(y_0 * y_0) = B(0) = M = \{0\}$. So $y * (x * (x * y)) = 0$. Thus $y \leq x * (x * y)$. Hence $x * (x * y) = y$. So X is medial. □

Let $(X, *, 0)$ be a generalized BCH-algebra. We now investigate conditions under which $Med(X)$ is a medial BCI-algebra.

Theorem 5.5. *Let $(X, *, 0)$ be a generalized BCH-algebra satisfying*

(11) $(x * (x * y)) * z = (x * z) * (x * y)$ *for all* $x, y, z \in X$.

Then $Med(X)$ is a medial BCI-algebra.

Proof. From Theorem 4.21, we get that $Med(X)$ is a subalgebra of X. To show that $Med(X)$ is a medial BCI-algebra, it is sufficient to show that $Med(X)$ satisfies (1), (1′) and (8).

Let $x, y \in Med(X)$. Since $Med(X)$ is a subalgebra, so $x*(x*y) \in Med(X)$. Now $x*(x*y) \leq y$ and $y \in Med(X)$. Hence $x*(x*y) = y$. So $Med(X)$ satisfies (8). Now $x*(x*y) = y$. This implies $(x*(x*y))*z = y*z$. Using (11) we get $(x*z)*(x*y) = y*z$ for all $x, y, z \in Med(X)$. Hence X satisfies (1). Further replacing y by $x*y$ in the last identity, we get $(x*z)*(x*(x*y)) = (x*y)*z$. Thus $(x*z)*y = (x*y)*z$, which gives X satisfies (1′). Hence $Med(X)$ is a medial BCI-algebra. □

Remark 5.6. *We note that a BCH-algebra $(X, *, 0)$ is a generalized BCH-algebra and it also satisfies* (11). *Thus the following result of Chaudhry* [3] *follows as a corollary of the above theorem.*

Corollary 5.7. *Let $(X, *, 0)$ be a BCH-algebra. Then $Med(X)$ is a medial BCI-algebra.*

Remark 5.8. *We note that if $(X, *, 0)$ is a generalized BCH-algebra satisfying* (11)*, then it contains at least two medial BCI-algebras namely $Med(X) = Min(X)$ and $\{0\}$.*

Conclusion

In this paper we have introduced a new class of algebras, the class of generalized BCH-algebras. This class is so general, that it contains the class of BCH-algebras, the class of BCI-algebras and the class of BCK-algebras.

We have initiated a study about a new class of abstract algebras, named as generalized BCH-algebras. The investigation of more properties and soft as well as fuzzy versions of generalized BCH-algebras and their ideals is a topic of further research and tremendous importance for researchers working in the field of algebraic structures, artificial intelligence and computing.

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