International Journal of Algebra, Vol. 11, 2017, no. 8, 381 - 387
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ija.2017.71051

Identification of $\mathfrak{C}_8$-Groups by Central Core

Mohammad Javad Ataei

Payame Noor University, P.O. Box 19395-3697 Tehran, Iran

Abstract

Let G denotes a semisimple $\mathfrak{C}_8$-group and $\{M_i \mid 1 \leq i \leq 8\}$ be a maximal irredundant 8-cover for G, with core-free intersection $D = \bigcap_{i=1}^{8} M_i$. Also for each i, $1 \leq i \leq 8$ we assume that $|G : M_i| = \alpha_i$ such that $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \alpha_5 \leq \alpha_6 \leq \alpha_7 \leq \alpha_8$.

Let G be a $\mathfrak{C}_n$-group, we say that G is a core i2-group if $(M_i)_G \cap (M_j)_G = 1$ for $3 \leq i < j \leq 8$. Also let G be a $\mathfrak{C}_n$-group and core i2-group, then we say that $N := (M_1)_G \cap (M_2)_G$ central core of G. In this paper we show that if G be a semisimple $\mathfrak{C}_8$-group and $\alpha_l \leq 4$ then G is core i-2 group. Also we give some results for central core of G.

Mathematics Subject Classification: 20F99

Keywords: covering groups by subgroups, cenral core, maximal iredundant cover, core-free intersection, core i-2 group

1 Introduction

Let G be a group. A set $\mathcal{C}$ of proper subgroups of G is called a cover for G if its set-theoretic union is equal to G. If the size of $\mathcal{C}$ is n, we call $\mathcal{C}$ an n-cover for the group G. A cover $\mathcal{C}$ for a group G is called irredundant if no proper subset of $\mathcal{C}$ is a cover for G. A cover $\mathcal{C}$ for a group G is called core-free if the intersection $D = \bigcap_{M \in \mathcal{C}} M$ of $\mathcal{C}$ is core-free in G, i.e. $D_G = \bigcap_{g \in G} g^{-1}Dg$ is the trivial subgroup of G. A cover $\mathcal{C}$ for a group G is called maximal if all the members of $\mathcal{C}$ are maximal subgroups of G. A cover $\mathcal{C}$ for a group G is called a $\mathfrak{C}_n$-cover whenever $\mathcal{C}$ is an irredundant maximal core-free n-cover for G and in this case we say that G is a $\mathfrak{C}_n$-group. A finite group is called semisimple if it has no non-trivial normal abelian subgroups (see p. 86 of [17] for further information on such groups).

Also we use the usual notations ([17]); for example, C_n denotes the cyclic group of order n, $(C_n)^j$ is the direct product of j copies of C_n, the core of a subgroup H of G is denoted by H_G.

In [18], Scorza determined the structure of all groups having an irredundant 3-cover with core-free intersection.

Theorem 1.1 (Scorza [18]) *Let $\{A_i : 1 \leq i \leq 3\}$ be an irredundant cover with core-free intersection D for a group G. Then $D = 1$ and $G \cong C_2 \times C_2$.*

In [15], Greco characterized all groups having an irredundant 4-cover with core-free intersection. Bryce et al.[13], characterized groups with maximal irredundant 5-cover with core-free intersection.

We characterized groups with maximal irredundant 6-cover with core-free intersection in [1], as below:

Theorem 1.2 *Let G be a group. Then G has a maximal irredundant 6-cover with core-free intersection D if and only if G satisfies one of the following properties.*

1. $|D| = 1$ *and* $G \cong C_5 \times C_5$;
2. $|D| = 1$ *and* $G \cong C_3 \times C_3 \times C_3$;
3. $|D| = 1$ *and* $G \cong Sym_3 \times Sym_3$;
4. $|D| = 1$ *and* $G \cong (C_3 \times C_3) \rtimes C_2$ *with* $Z(G) = 1$;
5. $|D| = 2$ *and* $G \cong (C_3)^3 \rtimes C_2$ *with* $Z(G) = 1$;
6. $|D| = 1$ *and* $G \cong C_2 \times C_2 \times Sym_3$ *or* $G \cong C_2 \times G_0$ *where* $G_0 = (C_3 \times C_3) \rtimes C_2$ *with* $Z(G_0) = 1$;
7. $|D| = 1$ *and* $G \cong C_5 \rtimes C_2$ *or* $G \cong C_5 \rtimes C_4$ *and* $Z(G) = 1$;
8. $|D| = 2$ *and* $G \cong (C_5 \times C_5) \rtimes C_2$ *with* $Z(G) = 1$;
9. $|D| = 4$ *and* $G \cong (C_5 \times C_5) \rtimes C_4$ *with* $Z(G) = 1$.

Abdollahi et al.[3], characterized groups with maximal irredundant 7-cover with core-free intersection.

Also we characterized p-groups with maximal irredundant 8-cover with core-free intersection in [2].

Theorem 1.3 (*See [2]*). *Let G be a $\mathfrak{C}_8$-group. Then G is a p-group for a prime number p if and only if $G \cong (C_3)^4$ or $(C_7)^2$.*

Also we investigated covering groups by subgroups and semisimplity condition in [4], covering semisimple groups by subgroups in [7], C_8-groups and nilpotency condition in [5], minimal normal subgroups and semisimplity condition in [8], characterization of 5-groups with a maximal irredundant 10-cover in [6], C_8 groups and subdirect product condition in [9] and subdirect product and covering groups by subgroups in [10]. Also we give some results on the number of C_8-groups for some primitive subgroups in [11]. Also we investigated some results on C_8-groups by index condition on maximal subgroups in [12].

Further problems of a similar nature, with slightly different aspects, have been studied by many people (see [16,19,20]).

2 Main Results

In this section we define core i2-groups and central core of G and give some results.

Definition 2.1 *Let G be a $\mathfrak{C}_n$-group, we say that G is a core i2-group if $(M_i)_G \cap (M_j)_G = 1$ for $3 \leq i < j \leq 8$.*

Definition 2.2 *Let G be a $\mathfrak{C}_n$-group and G is a core i2-group, then we say that $N := (M_1)_G \cap (M_2)_G$ central core of G.*

In the proofs of the main results we need the following lemmas:

Lemma 2.3 (*Lemma 2.2 of* [13]). *Let $\Gamma = \{A_i \ : \ 1 \leq i \leq m\}$ be an irredundant covering of a group G whose intersection of the members is D.*

(a) If p is a prime, x a p-element of G and $|\{i : x \in A_i\}| = n$, then either $x \in D$ or $p \leq m - n$.

(b) $\bigcap_{j \neq i} A_j = D$ for all $i \in \{1, 2, ..., m\}$.

(c) If $\bigcap_{i \in S} A_i = D$ whenever $|S| = n$, then $\left|\bigcap_{i \in T} A_i : D\right| \leq m - n + 1$ whenever $|T| = n - 1$.

(d) If Γ is maximal and U is an abelian minimal normal subgroup of G, then if $|\{i : U \subseteq A_i\}| = n$, either $U \subseteq D$ or $|U| \leq m - n$.

Lemma 2.4 (*Lemma 3.1 of* [20]). *Let M be a proper subgroup of the finite group G and let H_1, H_2, ..., H_k be subgroups with $|G : H_i| = \beta_i$ and $\beta_1 \leq \beta_2 \leq ... \leq \beta_k$. If $G = M \cup H_1 \cup \cdots \cup H_k$ then $\beta_1 \leq k$. Furthermore if $\beta_1 = k$ then $\beta_1 = \beta_2 = \cdots = \beta_k = k$ and $H_i \cap H_j \leq M$ for all $i \neq j$.*

Lemma 2.5 (*Lemma 3.2 of* [20]). *Let N be a normal subgroup of the finite group G. Let U_1,...,U_h be proper subgroups of G containing N and V_1,..., V_k be subgroups such that $V_iN = G$ with $|G : V_i| = \beta_i$ and $\beta_1 \leq \beta_2 \leq ... \leq \beta_k$. If $G = U_1 \bigcup \cdots \bigcup U_h \bigcup V_1 \bigcup \cdots \bigcup V_k$ then $\beta_1 \leq k$. Furthermore if $\beta_1 = k$ then $\beta_1 = \beta_2 = \cdots = \beta_k = k$ and $V_i \bigcap V_j \subseteq U_1 \bigcup \cdots \bigcup U_h$ for all $i \neq j$.*

Remark 2.6 *(1) The only primitive subgroups of degree 5 are C_5, $C_5 \rtimes C_2$, $C_5 \rtimes C_4$, Alt_5 and Sym_5.*

(2) The only primitive subgroups of degree 6 are Alt_5, Alt_6, Sym_5 and Sym_6.

(3) The only primitive subgroups of of degree 7 are C_7, $C_7 \rtimes C_2$, $C_7 \rtimes C_3$, $AGL(1,7)$, $PSL(3,2)$, Alt_7 and Sym_7.

Theorem 2.7 *If $\alpha_1 \leq \alpha_2 \leq 4$, then G is core i-2 group.*

Proof. Suppose, on the contrary, that G is not core i-2 group, then $(M_i)_G \cap (M_j)_G \neq 1$ for some $3 \leq i < j \leq 8$. Then there exists a minimal nontrivial normal subgroup U of G, such that $U \leq (M_i)_G \cap (M_j)_G$ for $3 \leq i < j \leq 8$.

On the other hand since $(M_1)_G \cap (M_2)_G \neq 1$, we have

$$\frac{G}{(M_1)_G \cap (M_2)_G} \hookrightarrow Sym_4 \times Sym_4.$$

Now if R is the CR-centerless radical of G, then

$$R = R^n \leqslant G^n \leqslant (M_1)_G \cap (M_2)_G.$$

Since $U \leqslant R$, therefore $U = 1$, which is a contradiction.

Theorem 2.8 *Let G be a semisimple $\mathfrak{C}_8$-group. If $\alpha_1 \leq \alpha_2 \leq 4$ and N central core of G, then $N \nsubseteq M_i$ for $3 \leq i \leq 8$.*

Proof. Suppose, on the contrary, that central core of G satisfy in $N \subseteq M_j$ for some $3 \leq j \leq 8$, say $j_\circ$. Then $N \subseteq (M_{j_\circ})_G$. Since $N \cap (M_{j_\circ})_G \cap (M_k)_G = 1$, for $3 \leq k \neq j_\circ \leq 8$, therefore $(M_1)_G \cap (M_2)_G \cap (M_k)_G = 1$, for $3 \leq k \neq j_\circ \leq 8$. Also by Theorem 2.7, we conclude that $j_\circ$ is unique. Therefore if $T := (M_1)_G \cap (M_2)_G \cap (M_{j_\circ})_G$, then $T \neq 1$. Therefore since $N \nsubseteq M_i$ for $3 \leq i \neq j_\circ \leq 8$ and $G = NM_i$, by Lemma 2.5, $\alpha_j \leq 5$ for $3 \leq j \neq j_\circ \leq 8$. Since $(M_1)_G \cap (M_2)_G \cap (M_j)_G = 1$, for $3 \leq j \neq j_\circ \leq 8$, for $3 \leq j \neq j_\circ \leq 8$, we have

$$G = \frac{G}{(M_1)_G \cap (M_2)_G \cap (M_j)_G} \hookrightarrow \frac{G}{(M_1)_G} \times \frac{G}{(M_2)_G} \times \frac{G}{(M_j)_G}.$$

Also $\frac{G}{(M_1)_G}$ and $\frac{G}{(M_2)_G}$ are primitive groups of degree at most 4, and $\frac{G}{(M_j)_G}$ is a primitive group of degree at most 5 for $3 \leq j \neq j_\circ \leq 8$.

We have used the following function written with GAP [14] program for all the subdirect products of $\{\frac{G}{(M_1)_G}, \frac{G}{(M_2)_G}, \frac{G}{(M_j)_G}\}$, for $3 \leq j \neq j_\circ \leq 8$. The output of this program is a list of groups.

```
 f3:=function(G,H,K) local S,M,T,R,Q,W,i,j;
M:=SubdirectProducts(G,H); S:=[]; for i in [1..Size(M)]  do
Add(S,SubdirectProducts(M[i],K)); od; T:=S; R:=[]; for i in
[1..Size(T)] do Add(R,T[i]); od; return R; Q:=R; W:=[]; for i in
[1..Size(Q)] do Add(W,List(Q[i],x->Size(RadicalGroup(x)))); od;
return W;end;
```

Now by the following command one can test each of these groups for being semisimple:

```
Size(RadicalGroup(W))
```

Then the group W is semisimple if and only if the output of this latter command is 1.

We obtain that the size of all the radical groups are nontrivial, which contradicts the semisimplity of G.

Now we introduce one question for researchers, because answer to bellow question is very important.

Question 2.9 *Let n are positive integer numbers, Are we classification $\mathfrak{C}_n$-group by central core?*

Acknowledgements. This research was supported by a grant from Payame Noor University.

References

[1] A. Abdollahi, M.J. Ataei, S.M. Jafarian Amiri, A. Mohammadi Hassanabadi, Groups with a maximal irredundant 6-cover, *Comm. Algebra,* **33** (2005), 3225-3238. https://doi.org/10.1081/agb-200066157

[2] A. Abdollahi, M.J. Ataei, A. Mohammadi Hassanabadi, Minimal blocking set in $PG(n, 2)$ and covering groups by subgroups, *Comm. Algebra,* **36** (2008), 365-380. https://doi.org/10.1080/00927870701715639

[3] A. Abdollahi, S.M. Jafarian Amiri, On groups with an irredundant 7-cover, *Journal of Pure and Applied Algebra,* **209** (2007), 291-300. https://doi.org/10.1016/j.jpaa.2006.05.021

[4] M.J. Ataei, Semisimplity Condition and Covering Groups by Subgroups, *International Journal of Algebra,* **4** (2010), no. 22, 1063-1068.

[5] M.J. Ataei, C_8-Groups and Nilpotency Condition, *International Journal of Algebra,* **4** (2010), no. 22, 1057-1062.

[6] M.J. Ataei, V. Sajjad, Characterization of 5-groups with a maximal Irredundant 10-cover, *International Mathematical Forum,* **6** (2011), no. 35, 1733-1738.

[7] M.J. Ataei, Covering semisimple groups by Subgroups, *International Journal of Algebra,* **5** (2011), no. 14, 661-665.

[8] M.J. Ataei, Miniimal Normal Subgorups and Semisimplity Condition, *International Journal of Algebra,* **6** (2012), no. 4, 179-183.

[9] M.J. Ataei, C_8-Groups and Subdirect Product Condition, *International Journal of Algebra,* **7** (2013), no. 14, 679-683. https://doi.org/10.12988/ija.2013.3435

[10] M.J. Ataei, Subdirect Product and Covering Groups by Subgroups, *International Journal of Algebra,* **7** (2013), no. 14, 673-677. https://doi.org/10.12988/ija.2013.3434

[11] M.J. Ataei, The Number of C_8-Groups for Some Primitive Subgroups, *International Journal of Algebra,* **8** (2014), no. 14, 681-685. https://doi.org/10.12988/ija.2014.4770

[12] M.J. Ataei, Investigation of C_8-Groups by Index Condition on Maximal Subgroups, *International Journal of Algebra,* **8** (2014), no. 14, 675-679. https://doi.org/10.12988/ija.2014.4769

[13] R.A. Bryce, V. Fedri and L. Serena, Covering groups with subgroups, *Bull. Austral. Math. Soc.*, **55** (1997), 469-476. https://doi.org/10.1017/s0004972700034109

[14] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.3*; 2002. http://www.gap-system.org

[15] D. Greco, Sui gruppi che sono somma di quattro o cinque sottogruppi, *Rend. Accad. Sci. Fis. Math. Napoli*, **23** (1956), no. 4, 49-59.

[16] B.H. Neumann, Groups covered by finitely many cosets, *Publ. Math. Debrecen*, **3** (1954), 227-242.

[17] D.J.S. Robinson, *A Course in the Theory of Groups,* Springer-Verlag, 1982. https://doi.org/10.1007/978-1-4684-0128-8

[18] G. Scorza, I gruppi che possono pensarsi come somma di tre loro sottogruppi, *Boll. Un. Mat. Ital.*, **5** (1926), 216-218.

[19] M.J. Tomkinson, Groups covered by finitely many cosets or subgroups, *Comm. Algebra*, **15** (1987), 845-859. https://doi.org/10.1080/00927878708823445

[20] M.J. Tomkinson, Groups as the union of proper subgroups, *Math. Scand.* **81** (1997), 191-198. https://doi.org/10.7146/math.scand.a-12873

Received: October 28, 2017; Published: December 11, 2017