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The $\mathfrak{C}_8$-Group Having Five Maximal Subgroups of Index 2 and Three of Index 3

Mohammad Javad Ataei

Payame Noor University, P.O. Box 19395-3697 Tehran, Iran

Abstract

A cover $\mathcal{C}$ for a group G is called a $\mathfrak{C}_n$-cover whenever $\mathcal{C}$ is an irredundant maximal core-free n-cover for G and in this case we say that G is a $\mathfrak{C}_n$-group. In this paper we prove that the only $\mathfrak{C}_8$-group having five maximal subgroups of index 2 and three of index 3 occuring as a subdirect products of three C_2s and two primitive groups of degree 3, is (isomorphic to) $C_2 \times C_2 \times ((C_3 \times C_3) \rtimes C_2)$ for which $D = 1$.

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1 Introduction

Let G be a group. A set $\mathcal{C}$ of proper subgroups of G is called a cover for G if its set-theoretic union is equal to G. If the size of $\mathcal{C}$ is n, we call $\mathcal{C}$ an n-cover for the group G. A cover $\mathcal{C}$ for a group G is called irredundant if no proper subset of $\mathcal{C}$ is a cover for G. A cover $\mathcal{C}$ for a group G is called core-free if the intersection $D = \bigcap_{M\in\mathcal{C}} M$ of $\mathcal{C}$ is core-free in G, i.e. $D_G = \bigcap_{g\in G} g^{-1}Dg$ is the trivial subgroup of G. A cover $\mathcal{C}$ for a group G is called maximal if all the members of $\mathcal{C}$ are maximal subgroups of G. A cover $\mathcal{C}$ for a group G is called a $\mathfrak{C}_n$-cover whenever $\mathcal{C}$ is an irredundant maximal core-free n-cover for G and in this case we say that G is a $\mathfrak{C}_n$-group. A finite group is called semisimple if it has no non-trivial normal abelian subgroups (see p. 86 of [17] for further information on such groups).

Also we use the usual notations ([17]); for example, C_n denotes the cyclic group of order n, $(C_n)^j$ is the direct product of j copies of C_n, the core of a subgroup H of G is denoted by H_G.

In [18], Scorza determined the structure of all groups having an irredundant 3-cover with core-free intersection.

Theorem 1.1 (Scorza [18]) *Let $\{A_i : 1 \le i \le 3\}$ be an irredundant cover with core-free intersection D for a group G. Then $D = 1$ and $G \cong C_2 \times C_2$.*

In [15], Greco characterized all groups having an irredundant 4-cover with core-free intersection. Bryce et al.[13], characterized groups with maximal irredundant 5-cover with core-free intersection.

We characterized groups with maximal irredundant 6-cover with core-free intersection in [1], as below:

Theorem 1.2 *Let G be a group. Then G has a maximal irredundant 6-cover with core-free intersection D if and only if G satisfies one of the following properties.*

1. $|D| = 1$ *and* $G \cong C_5 \times C_5$;
2. $|D| = 1$ *and* $G \cong C_3 \times C_3 \times C_3$;
3. $|D| = 1$ *and* $G \cong Sym_3 \times Sym_3$;
4. $|D| = 1$ *and* $G \cong (C_3 \times C_3) \rtimes C_2$ *with* $Z(G) = 1$;
5. $|D| = 2$ *and* $G \cong (C_3)^3 \rtimes C_2$ *with* $Z(G) = 1$;
6. $|D| = 1$ *and* $G \cong C_2 \times C_2 \times Sym_3$ *or* $G \cong C_2 \times G_0$ *where* $G_0 = (C_3 \times C_3) \rtimes C_2$ *with* $Z(G_0) = 1$;
7. $|D| = 1$ *and* $G \cong C_5 \rtimes C_2$ *or* $G \cong C_5 \rtimes C_4$ *and* $Z(G) = 1$;
8. $|D| = 2$ *and* $G \cong (C_5 \times C_5) \rtimes C_2$ *with* $Z(G) = 1$;
9. $|D| = 4$ *and* $G \cong (C_5 \times C_5) \rtimes C_4$ *with* $Z(G) = 1$.

Abdollahi et al.[3], characterized groups with maximal irredundant 7-cover with core-free intersection.

Also we characterized p-groups with maximal irredundant 8-cover with core-free intersection in [2].

Theorem 1.3 (*See [2]*). *Let G be a $\mathfrak{C}_8$-group. Then G is a p-group for a prime number p if and only if $G \cong (C_3)^4$ or $(C_7)^2$.*

Also we investigated covering groups by subgroups and semisimplity condition in [4], covering semisimple groups by subgroups in [7], C_8-groups and nilpotency condition in [5], minimal normal subgroups and semisimplity condition in [8], characterization of 5-groups with a maximal irredundant 10-cover in [6], C_8 groups and subdirect product condition in [9] and subdirect product and covering groups by subgroups in [10]. Also we give some results on the number of C_8-groups for some primitive subgroups in [11]. Also we investigated some results on C_8-groups by index condition on maximal subgroups in [12].

Further problems of a similar nature, with slightly different aspects, have been studied by many people (see [16,19,20]).

Let D denote the intersection of an arbitrary maximal irredundant 8-cover with core-free intersection. In this paper we prove that the only $\mathfrak{C}_8$-group having five maximal subgroups of index 2 and three of index 3 occuring as a subdirect products of three C_2s and two primitive groups of degree 3, is (isomorphic to) $C_2 \times C_2 \times ((C_3 \times C_3) \rtimes C_2)$ for which $D = 1$.

2 Main Results

To obtain the following results using GAP; we had use several computers mostly for a long time.

Theorem 2.1 *The only $\mathfrak{C}_8$-group having five maximal subgroups of index 2 and three of index 3 occuring as a subdirect products of three C_2s and two primitive groups of degree 3, is (isomorphic to)*

$C_2 \times C_2 \times ((C_3 \times C_3) \rtimes C_2)$ *for which $D = 1$.*

Proof. We have used the following function written in GAP [14] to prove this theorem. The input of the function is a group G and the output are all irredundant 8-covers with core-free intersection of G five of whose members have index 2 and three of whose members have index 3, and if there is no such cover for G, the output is the empty list.

```
gfe3c22s35index23index3:=function(G,k) local
S,M,n,i,j,t,T,Q,R,F2,F3,C1,C2; n:=Size(G); M:=MaximalSubgroups(G);
F2:=Filtered(M,i->Index(G,i)=2); C1:=Combinations(F2,5);
F3:=Filtered(M,i->Index(G,i)=3); C2:=Combinations(F3,3); S:=[];
for i in [1..Size(C1)] do for  j in [1..Size(C2)]  do if
Size(Union(Union(C1[i]),Union(C2[j])))=n then
Add(S,Union(C1[i],C2[j]));fi;od;od; T:=[]; for i in [1..Size(S)]
do if Size(Core(G,Intersection(S[i])))=1 then Add(T,S[i]); fi; od;
R:=[]; for i in [1..Size(T)] do Q:=Combinations(T[i],k-1); if (n
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```
in List(Q,i->Size(Union(i))))=false then Add(R,T[i]); fi; od;
return R; end;
```

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