**International Journal of Algebra, Vol. 11, 2017, no. 5, 225 - 230**
**HIKARI Ltd, www.m-hikari.com**
**https://doi.org/10.12988/ija.2017.7522**

# On the Diophantine Equation $8^x + 113^y = z^2$

**Shivangi Asthana**

Department of Mathematics
North-Eastern Hill University
Shillong-793022, Meghalaya, India

**Madan Mohan Singh**

Department of Basic Sciences and Social Sciences
North - Eastern Hill University
Shillong-793022, Meghalaya, India



**Abstract**

In this paper we have shown that the Diophantine equation $8^x + 113^y = z^2$ has exactly three non-negative integer solutions for $x$, $y$ and $z$ . The solutions are $(1, 0, 3)$, $(1, 1, 11)$ and $(3, 1, 25)$ respectively.

**Mathematics Subject Classification:** 11D61

**Keywords:** Exponential Diophantine equation

# 1 Introduction

There is a lot of studies about Diophantine equations of the type $a^x + b^y = c^z$ by a number of mathematicians in the field of number theory. In 1999, Cao [1], proved that this equation has at most one solution with $c > 1$. In 2012, Peker and Cenberci [2] suggested that the Diophantine equation $8^x + 19^y = z^2$ has no non-negative integer solution. However, in the same year, Sroysang [3] proved that the Diophantine equation $8^x + 19^y = z^2$ has a unique non-negative integer solution in $(x, y, z)$ which is $(1, 0, 3)$. Same author [4, 5] also solved the Diophantine equations $8^x + 13^y = z^2$ and $8^x + 7^y = z^2$, respectively. He found that the solution of these equations is $(1, 0, 3)$ in non-negative integers $(x, y, z)$. Rabago [6] showed that the Diophantine equation $8^x + 17^y = z^2$ has four solutions in non-negative integers $(x, y, z)$. Further, several Diophantine equations of different types have been studied by different workers [7, 8, 9, 10, 11, 12, 13]. Recently, Qi and Li [14] established that the Diophantine equation $8^x + p^y = z^2$, $x$, $y$, $z$ belong to natural number and $p$ is an odd integer, with $p \equiv 1(mod\ 8)$ and $p \neq 17$, have at most two positive integer solutions in $(x, y, z)$ where $p$ is an odd prime. Hence, it is a matter of further investigation to examine that, apart from $p = 17$, how many other such Diophantine equations are there which do not obey Qi and Li's [14] generalization. Although a number of other Diophantine equations has been solved by several other authors, yet it is imperative to search many more Diophantine equations violating Qi and Li's generalization [14], we have made an attempt to solve the new Diophantine equation containing $p = 113$, hitherto uninvestigated by any researcher to the best of our knowledge, and have found that it has three exact solutions in non-negative integers $(x, y, z)$. This problem constitutes second exception, apart from that of the Diophantine equation $8^x + 17^y = z^2$ by Rabago[6] to the Qi and Li's generalizations regarding solutions of Diophantine equations of the type $8^x + p^y = z^2$ with $p \equiv 1(mod\ 8)$ where $p$ is a prime.

# 2 Preliminaries

The Catalan's conjecture is an important well known conjecture and plays an important role in solving Diophantine equations. According to this conjecture, $(3, 2, 2, 3)$ is a unique solution $(a, x, b, y)$ for the Diophantine equation $a^x - b^y = 1$ where $a$, $b$, $x$ and $y$ are integers with $\min\{a, b, x, y\} > 1$. This conjecture was proved by Mihailescu in 2004 [15].

## 2.1 Proposition

$(3, 2, 2, 3)$ is a solution $(a, b, x, y)$ for the Diophantine equation $a^x - b^y = 1$ where $a$, $b$, $x$ and $y$ are integers with $\min\{a, b, x, y\} > 1$.
Now we will prove two Lemma's by Proposition 2.1.

**Lemma 2.1.** $(1, 3)$ *is a unique solution* $(x, z)$ *for the Diophantine equation* $8^x + 1 = z^2$ *where* $x$ *and* $z$ *are non-negative integers.*

*Proof.* Let $x$ and $z$ be non-negative integers such that $8^x + 1 = z^2$. First we consider the case $x = 0$ and $z = 0$. If $x = 0$, then $z^2 = 2$ which is impossible. If $z = 0$, then $8^x = -1$ which is impossible. Now we consider the case $x, z > 0$. Then $8^x + 1 = z^2$ or $8^x = z^2 - 1$. Then $2^{3x} = (z - 1)(z + 1)$. Thus, $(z - 1) = 2^u$ where $u$ is a non-negative integer. Then $(z + 1) = 2^{3x-u}$. Thus, $2 = 2^{3x-u} - 2^u$ or $2^u(2^{3x-2u} - 1) = 2$. We have two possibilities

i) $2^u = 2^0$ which implies that $u = 0$ and $2^{3x} - 1 = 2$ which implies that $2^{3x} = 3$, which is impossible.
ii) $2^u = 2$ which implies that $u = 1$ and $2^{3x-2} - 1 = 1$ or $2^{3x-2} = 2$ which implies that $x = 1$.
Putting $x = 1$ in $8^x + 1 = z^2$, we have $z = 3$. Hence, $(1, 3)$ is a unique solution $(x, z)$ for the Diophantine equation $8^x + 1 = z^2$ where $x$ and $z$ are non-negative integers. □

**Lemma 2.2.** *The Diophantine equation* $1 + 113^y = z^2$ *has no non-negative integer solution.*

*Proof.* Suppose that there are non-negative integers $y$ and $z$ such that $1 + 113^y = z^2$. If $y = 0$, then $1 + 1 = z^2$ i.e $z^2 = 2$ which is impossible. If $z = 0$, then $113^y = -1$ which is not possible. We consider the case when $y, z > 0$. Then $z^2 = 1 + 113^y$ or $113^y = z^2 - 1 = (z - 1)(z + 1)$. Thus, $z - 1 = 113^v$, where $v$ is a non-negative integer. Then $z + 1 = 113^{y-v}$. Thus, $2 = 113^{y-v} - 113^v = 113^v(113^{y-v} - 1)$ which implies that $v = 0$ and $113^y - 1 = 2$ or $113^y = 3$ which is not possible. Hence, the Diophantine equation $1 + 113^y = z^2$ has no non-negative integer solution. □

# 3 Main Result

**Theorem 3.1.** *The Diophantine equation* $8^x + 113^y = z^2$ *has exactly three solutions in non-negative integers* $(x, y, z) \in \{(1, 0, 3), (1, 1, 11), (3, 1, 25)\}$.

*Proof.* Let $x$, $y$ and $z$ be non-negative integers such that $8^x + 113^y = z^2$. We first consider the case when $y$ is zero. By Lemma 2.1, we have $(x, y, z) = (1, 0, 3)$. From Lemma 2.2, $x \geq 1$. This implies that $z$ is odd.
Now we will divide $y$ into two cases when $x \geq 1$.
Case(i) If $y$ is even i.e. $y = 2l$ for some positive integer $l$, then $8^x = z^2 - 113^{2l} = (z - 113^l)(z + 113^l)$ or, $2^{3x} = (z - 113^l)(z + 113^l)$. This implies that $2 \cdot 113^k = 2^w(2^{3x-2w} - 1)$ where $z - 113^l = 2^w$ and $z + 113^l = 2^{3x-w}$, $w$ is a non-negative integer. We have two subcases:
a) $w = 0$. Then $z - 113^l = 1$. This implies that $z$ is even. This is a contradiction.
b) $w = 1$. Then $2^{3x-2} - 1 = 113^k$. Then $2^{3x-2} - 113^l = 1$. If $x = 1$, then $113^l = 1$ i.e $l = 0$ so $y = 0$. Thus, $x \geq 2$. By Proposition 2.1, we have $l = 1$. Then $2^{3x-2} = 114$. This is impossible.
Case(ii) When $y$ is odd. Let $y = 2l + 1$ where $l$ is a non-negative integer. We will divide this case into two parts i.e. Part(1) and Part(2).
Part(1) $8^x + 113^{2l+1} = z^2$
or, $8^x + (13 + 100) \cdot 113^{2l} = z^2$
or, $8^x + 13 \cdot 113^{2l} = z^2 - 100 \cdot 113^{2l}$

or, $8^x + 13 \cdot 113^{2l} = (z - 10 \cdot 113^l)(z + 10 \cdot 113^l)$
There are two possibilities for this equation

$$\begin{cases} z - 10 \cdot 113^l = 1 \\ z + 10 \cdot 113^l = 8^x + 13 \cdot 113^{2l} \end{cases}$$

or

$$\begin{cases} z + 10 \cdot 113^l = 1 \\ z - 10 \cdot 113^l = 8^x + 13 \cdot 113^{2l} \end{cases}$$

Solving first set of equalities, we have $113^l(20 - 13 \cdot 113^l) = 8^x - 1$. It implies that $l = 0$ and $20 - 13 = 8^x - 1$. Hence we obtain $x = 1$, $y = 1$ and $z = 11$. Hence, the solution of the Diophantine equation in non-negative integers $(x, y, z) = (1, 1, 11)$. Solving second set of equalities, we have $113^l(20 - 13 \cdot 113^l) = 1 - 8^x$ which implies that $l = 0$ and $20 - 13 = 1 - 8^x$ or $-8^x = 6$ which is not solvable.

Part(2) Again we have, $8^x + 113^{2l+1} = z^2$
or, $8^x + 113 \cdot 113^{2l} = z^2$
or, $8^x + (576 - 463) \cdot 113^{2l} = z^2$
or, $8^x - 463 \cdot 113^{2l} = z^2 - 576 \cdot 113^{2l}$
or, $8^x - 463 \cdot 113^{2l} = (z - 24 \cdot 113^l)(z + 24 \cdot 113^l)$
There are two possibilities for this equation

$$\begin{cases} z - 24 \cdot 113^l = 1 \\ z + 24 \cdot 113^l = 8^x - 463 \cdot 113^{2l} \end{cases}$$

or

$$\begin{cases} z + 24 \cdot 113^l = 1 \\ z - 24 \cdot 113^l = 8^x - 463 \cdot 113^{2l} \end{cases}$$

Solving first set of equalities, we have $113^l(48 + 463 \cdot 113^l) = 8^x - 1$. It implies that $l = 0$ and $48 + 463 = 8^x - 1$ or $511 + 1 = 8^x$ or $512 = 8^x$ or $8^x = 8^3$ which implies that $x = 3$. Hence we obtain $x = 3$, $y = 1$ and $z = 25$. Therefore the solution of this Diophantine equation is $(3, 1, 25)$. Solving second set of equalities, we have $113^l(48 - 463 \cdot 113^l) = 1 - 8^x$. It implies that $l = 0$ and $48 - 463 = 1 - 8^x$ or $-416 = -8^x$ or $416 = 8^x$ which is not solvable. □

Corollary 3.2 The Diophantine equation $8^x + 113^y = w^4$ has a unique non-negative integer solution $(x, y, z) = (3, 1, 5)$.

*Proof.* Suppose that there are non-negative integers $x$, $y$ and $w$ such that $8^x + 113^y = w^4$. Let $z = w^2$. Then $8^x + 113^y = z^2$. By Theorem 3.1,we have $(x, y, z) = (3, 1, 25)$. Then $w^2 = z = 25$ i.e $w = 5$. Hence, the equation $8^x + 113^y = w^4$ has unique solution in non-negative integers $(x, y, w) = (3, 1, 5)$. □

# 4 Conclusion

In this Theorem, we have solved the Diophantine equation $8^x + 113^y = z^2$ where 113 is a prime number. We have shown that the entitled equation has three non-negative integer solutions in $(x, y, z)$ i.e. $(1, 0, 3)$, $(1, 1, 11)$ and $(3, 1, 25)$. Thus, we have proved that the entitled equation offers an exception to the generalization made by Li and Qi that the prime numbers satisfying the congruence equation $p \equiv 1(mod\ \ 8)$ have at most two positive integer solutions $(x, y, z)$ only.

# References

[1] Z. Cao, A note on the Diophantine equation $a^x + b^y = c^z$, *Acta Arith.*, **XCI** (1999), no. 1, 85-89.

[2] B. Peker, S. Cenberci, Solutions of the Diophantine equation $4^x+p^y = z^{2n}$, *Selcuk J. Appl. Math.*, **13** (2012), no. 2, 31-34.

[3] B. Sroysang, More on the Diophantine equation $8^x + 19^y = z^2$, *Int. J. Pure. Appl. Maths.*, **81** (2012), 601-604.

[4] B. Sroysang, On the Diophantine equation $8^x + 13^y = z^2$, *Int. J. Pure. Appl. Maths,* **90** (2014), no. 1, 69-72. https://doi.org/10.12732/ijpam.v90i1.9

[5] B. Sroysang, On the Diophantine equation $7^x + 8^x = z^2$, *Int. J. Pure. Appl. Maths,* **84** (2013), 111-114. https://doi.org/10.12732/ijpam.v84i1.8

[6] J. F. T. Rabago, On an open problem by B. Sroysang, *Konuralp Journal of Mathematics,* **1** (2013), no. 2, 30-32.

[7] A. Suvarnamani, On a Diophantine equation $2^x + p^y = z^2$, *Int. J. Math. Sci. Appl.,* **1** (2011), 1415-1419.

[8] T. Hadano, On the Diophantine equation $a^x = b^y + c^z$, *Math. J. Okayama Univ.,* **19** (1976), 25-29.

[9] J. F. T. Rabago, On two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$, *Int. J. of Math. and Scientific Computing,* **3** (2013), 28-29.

[10] B. Sroysang, On the Diophantine Equation $3^x + 5^y = z^2$, *International J. Pure. Appl. Maths.,* **81** (2012), 605-608.

[11] S. Chotchaisthit, On the Diophantine equation $2^x+11^y = z^2$, *Maejo Int. J. Sci. Technol.,* **7** (2013), 291-293. https://doi.org/10.14456/mijst.2013.24

[12] S. Chotchaisthit, On the Diophantine equation $p^x + (p+1)^y = z^2$, *Int. J. Pure. Appl. Maths.,* **88** (2013), 169-172.
https://doi.org/10.12732/ijpam.v88i2.2

[13] D. Acu, On the Diophantine equations of the type $a^x + b^y = c^z$, *Gen. Math.,* **13** (2005), 67-72.

[14] L. Qi, X. Li, The Diophantine equation $8^x + p^y = z^2$, *The Scientific World Journal,* **2015** (2015), Article ID 306590, 1-3.
https://doi.org/10.1155/2015/306590

[15] P. Mihailescu, Primary cyclotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.,* **2004** (2004), 167-195.
https://doi.org/10.1515/crll.2004.048

**Received: May 15, 2017; Published: June 29, 2017**