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On Generalized Derivations and Commutativity of Prime Rings with Involution

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Abstract

Let $\mathscr{R}$ be a ring with involution $'*'$. A map δ of the ring $\mathscr{R}$ into itself is called a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathscr{R}$. An additive map $\mathfrak{F} : \mathscr{R} \to \mathscr{R}$ is called a generalized derivation on $\mathscr{R}$ if $\mathfrak{F}(xy) = \mathfrak{F}(x)y + x\delta(y)$ for all $x, y \in \mathscr{R}$,

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where δ is a derivation of $\mathscr{R}$. In [1, Theorem 2.2], we proved that if a prime ring $\mathscr{R}$ with involution$'*'$ of the second kind and $char(\mathscr{R}) \neq 2$ admits a nonzero generalized derivation $\mathfrak{F}$ such that $\mathfrak{F}([x, x^*]) = 0$ for all $x \in \mathscr{R}$, then $\mathscr{R}$ is commutative. In fact, the proof of above mentioned result and [1, Theorem 2.5] are complicated and technical. The aim of this manuscript is to give a brief and elegant proofs of these results. As an application, and apart from proving the other results, many known theorems can be either generalized or deduced.

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1. Introduction

This study has been motivated by the various results proved by some well known algebraists(viz.; [6], [7], [9], [17] and [18]). Throughout the discussion, unless otherwise stated, $\mathscr{R}$ always denotes an associative ring with centre $Z(\mathscr{R})$. For each $s, t \in \mathscr{R}$, let $[s, t]$ denote the commutator $st - ts$ and the symbol $s \circ t$ will denote the anti-commutator $st + ts$, respectively. Given an integer $n \geq 2$, a ring R is said to be n-torsion free if $nx = 0$ (where $x \in \mathscr{R}$) implies that $x = 0$. A ring $\mathscr{R}$ is called prime if $a\mathscr{R}b = (0)$ (where $a, b \in \mathscr{R}$) implies $a = 0$ or $b = 0$, and is called semiprime ring if $a\mathscr{R}a = (0)$ (where $a \in \mathscr{R}$) implies $a = 0$. An additive map $x \mapsto x^*$ of $\mathscr{R}$ into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ hold for all $x, y \in \mathscr{R}$. A ring equipped with an involution is called ring with involution or $*$-ring. An element x in a ring with involution is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of $\mathscr{R}$ will be denoted by $H(\mathscr{R})$ and $S(\mathscr{R})$, respectively. The involution is called of the first kind if $\mathcal{Z}(\mathscr{R}) \subseteq H(\mathscr{R})$, otherwise it is said to be of the second kind. In the later case $S(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R}) \neq (0)$. Notice that in case x is normal i.e., $xx^* = x^*x$, if and only if h and k commute(see [15] for more details).

An additive mapping $\delta : \mathscr{R} \to \mathscr{R}$ is said to be a derivation on $\mathscr{R}$ if $\delta(st) = \delta(s)t + s\delta(t)$ for all $s, t \in R$. A derivation δ is said to be inner if there exists $a \in \mathscr{R}$ such that $\delta(s) = as - sa$ for all $s \in \mathscr{R}$. Following [11], an additive map $\mathfrak{F} : \mathscr{R} \to \mathscr{R}$

is called a generalized derivation on $\mathscr{R}$ if $\mathfrak{F}(xy) = \mathfrak{F}(x)y + x\delta(y)$ for all $x, y \in \mathscr{R}$, where δ is a derivation on $\mathscr{R}$. It is to remark that if an associated derivation δ is a nonzero, then generalized derivation $\mathfrak{F}$ must be nonzero. The familiar examples of generalized derivations are derivations and generalized inner derivations that is, the map of the form $\mathfrak{F} : \mathscr{R} \to \mathscr{R}$ such that $\mathfrak{F}(x) = ax + xb$ for all $x \in \mathscr{R}$(where a and b are fixed elements of $\mathscr{R}$.) Moreover, every map of the form $\mathfrak{F}(x) = ax + \delta(x)$ for all $x \in \mathscr{R}$(where a is an fixed element of $\mathscr{R}$ and δ is a derivation on $\mathscr{R}$) is a generalized derivation. Further, if $\mathscr{R}$ has 1, then all generalized derivations have the above mentioned form. One may observe that the concept of generalized derivation includes the concept of derivations, also of the left multipliers *i.e.*, an additive maps $\mathfrak{F} : \mathscr{R} \to \mathscr{R}$ such that $\mathfrak{F}(xy) = \mathfrak{F}(x)y$ for all $x, y \in \mathscr{R}$(when $\delta = 0$). In literature it is commonly knows as left centralizers (see [4] where further references can be looked. Hence it should be interesting to extend some results concerning these notions to generalized derivations. Thus, it is natural to ask what we can say about the commutativity of $\mathscr{R}$ if the derivation δ is replaced by a generalized derivation $\mathfrak{F}$. Some recent results were shown on generalized derivations in the following papers [1], [6], [7], [10], [12], [14] and [19] where further references can be found.

In this paper, our aim is to continue this line of investigation and discuss the commutativity of prime rings with involution involving generalized derivations. In particular, we extended some results proved in [2], [4], [5], [8] and [13] for derivations to generalized derivations.

2. Preliminaries

In this section, we collect some well known facts and results in rings which will be used frequently without specific mentioned.

Fact 2.1. *For all $s, t, r \in \mathscr{R}$; we have*

$$[st, r] = s[t, r] + [s, r]t \text{ and } [s, tr] = t[s, r] + [s, t]r$$

$$so(tr) = (sot)r - t[s, r] = t(sor) + [s, t]r \text{ and } (st)or = s(tor) - [s, r]t = (sor)t + s[t, r].$$

Fact 2.2. *[2, Fact 5] Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$. Let δ be a derivation of $\mathscr{R}$ such that $\delta(h) = 0$ for all $h \in H(\mathscr{R}) \cap \mathcal{Z}(\mathscr{R})$. Then $\delta(x) = 0$ for all $x \in \mathscr{R}$.*

Fact 2.3 ([3, Lemma 2.1]). *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$. If $\mathscr{R}$ is normal i.e., $[x, x^*] = 0$ for all $x \in \mathscr{R}$, then $\mathscr{R}$ is commutative.*

Fact 2.4. *The center of a prime ring is free from zero divisors.*

Fact 2.5. *Let $\mathscr{R}$ be a 2-torsion free ring with involution $'*'$. Then every $x \in \mathscr{R}$ can be uniquely represented as $2x = h + k$, where $h \in H(\mathscr{R})$ and $k \in S(\mathscr{R})$.*

3. Main Results

We facilitate our discussion with the following theorem which generalized many know results proved in [4], [5] and [13]. Precisely, first we give a brief proof of Theorems 2.2 & 2.5 of [1].

Theorem 3.1. *Let $\mathscr{R}$ be a prime ring with involution$'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$. Let $\mathfrak{F}$ be a generalized derivation of $\mathscr{R}$ such that $\mathfrak{F}([x, x^*]) = 0$ for all $x \in \mathscr{R}$. Then $\mathscr{R}$ is commutative.*

Proof. We are given that $\mathfrak{F} : \mathscr{R} \to \mathscr{R}$ a generalized derivation with an associated nonzero derivation $\delta : \mathscr{R} \to \mathscr{R}$ such that .

$$\mathfrak{F}([x, x^*]) = 0 \tag{3.1}$$

for all $x \in \mathscr{R}$. Linearization of relation (3.1) yields

$$\mathfrak{F}([x, y^*]) + \mathfrak{F}([y, x^*]) = 0 \tag{3.2}$$

for all $x, y \in \mathscr{R}$. Substituting yh for y (where $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$ in (3.2) and using the Fact 2.1 and $[x, h] = 0 = [h, x^*]$ for all $x \in \mathscr{R}$, we obtain

$$\mathfrak{F}([x, y^*]h) + \mathfrak{F}([y, x^*]h) = 0$$

for all $x, y \in \mathscr{R}$. Since $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$, so $\delta(h) \in \mathcal{Z}(\mathscr{R})$, we arrive at

$$\mathfrak{F}([x, y^*])h + [x, y^*]\delta(h) + \mathfrak{F}([y, x^*])h + [y, x^*]\delta(h) = 0$$

for all $x, y \in \mathscr{R}$. The above equation can be rewritten as

$$(\mathfrak{F}([x, y^*]) + \mathfrak{F}([y, x^*])h + ([x, y^*]) + [y, x^*])\delta(h) = 0 \tag{3.3}$$

for all $x, y \in \mathscr{R}$. Combining equations (3.2) and (3.3), we obtain

$$([x, y^*] + [y, x^*])\delta(h) = 0 \tag{3.4}$$

for all $x, y \in \mathscr{R}$. Now putting $x = y$ in (3.4), we get $2[x, x^*])\delta(h) = 0$ for all $x \in \mathscr{R}$. As $char(\mathscr{R}) \neq 2$, the last relation gives $[x, x^*]\delta(h) = 0$ for all $x \in \mathscr{R}$. Since $\delta(h) \in \mathcal{Z}(\mathscr{R})$, so the last expression gives $[x, x^*]r\delta(h) = 0$ for all $x, r \in \mathscr{R}$. This implies that $[x, x^*]\mathscr{R}\delta(h) = (0)$ for all $x, r \in \mathscr{R}$. By the primeness of $\mathscr{R}$, we conclude that either $[x, x^*] = 0$ for all $x \in \mathscr{R}$ or $\delta(h) = 0$ for all $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$. If $\delta(h) = 0$ for all $h \in \mathcal{Z}(\mathscr{R}) \bigcap H(\mathscr{R})$. Replacing h by k^2 (where $k \in \mathcal{Z}(\mathscr{R}) \cap S(\mathscr{R})$) in the last expression, we get $2\delta(k)k = 0$ for all $k \in \mathcal{Z}(\mathscr{R}) \bigcap S(\mathscr{R})$. Since $char(\mathscr{R}) \neq 2$, we arrive at $\delta(k)k = 0$ for all $k \in \mathcal{Z}(\mathscr{R}) \cap S(\mathscr{R})$. Since $k \in \mathcal{Z}(\mathscr{R}) \cap S(\mathscr{R})$ and $\mathscr{R}$ is prime, so by Fact 2.4 we conclude that $\delta(k) = 0$ for all $k \in \mathcal{Z}(\mathscr{R}) \cap S(\mathscr{R})$. In view Fact 2.5, for every $x \in \mathscr{R}$, we write $2x = h + k$, where $h \in H(\mathscr{R}),\ k \in S(\mathscr{R})$, since $char(\mathscr{R}) \neq 2$. This gives $2\delta(x) = \delta(2x) = \delta(h + k) = \delta(h) + \delta(k) = 0$ and hence $\delta(x) = 0$ for all $x \in \mathscr{R}$, a contradiction (see also Fact 2.2). Consequently, the remaining case is that $[x, x^*] = 0$ for all $x \in \mathscr{R}$, then the application of Fact 2.3 yields the required conclusion. Hence, $\mathscr{R}$ is commutative. This completes the proof of the theorem. $\square$

We now prove the next theorem in same domain.

Theorem 3.2. *Let $\mathscr{R}$ be a prime ring with involution $'*'$ of the second kind such that $char(\mathscr{R}) \neq 2$. Let $\mathfrak{F}$ be a generalized derivation of $\mathscr{R}$ such that $\mathfrak{F}(x \circ x^*) = 0$ for all $x \in \mathscr{R}$. Then $\mathscr{R}$ is commutative.*

Proof. Direct linearization of given assumption yields

$$\mathfrak{F}(x \circ y^*) + \mathfrak{F}(y \circ x^*) = 0 \tag{3.5}$$

for all $x, y \in \mathscr{R}$. Replacing y by yh(where $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$ in (3.5) and using the anti-commutator identities $x \circ (y^*h) = (x \circ y^*)h - y^*[x, h]$ and $(yh) \circ (x^*) = (y \circ x^*)h + y[h, x^*]$, we get

$$\mathfrak{F}((x \circ y^*)h) + \mathfrak{F}((y \circ x^*)h) = 0$$

for all $x, y \in \mathscr{R}$ and $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$. The above expression gives

$$\mathfrak{F}\{(x \circ y^*) + \mathfrak{F}(y \circ x^*)\}h + \{(x \circ y^*) + (y \circ x^*)\}\delta(h) = 0 \tag{3.6}$$

for all $x, y \in \mathscr{R}$ and $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$. In view of given hypothesis, we arrive at

$$([x, y^*] + [y, x^*])\delta(h) = 0 \tag{3.7}$$

for all $x, y \in \mathscr{R}$ and $h \in \mathcal{Z}(\mathscr{R}) \cap H(\mathscr{R})$. The last equation is same as equation (3.4). Henceforth, using the same arguments as we have used in the proof of the last paragraph of Theorem 3.1, we get the required result. This proves the theorem. □

Theorem 3.3. *Let $\mathscr{R}$ be a prime ring with involution$'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$. Let $\mathfrak{F}$ be a generalized derivation of $\mathscr{R}$. Then the following conditions are mutually equivalent:*

(*i*) $\mathfrak{F}([x, x^*]) + [x, x^*] = 0$ *for all* $x \in \mathscr{R}$;
(*ii*) $\mathfrak{F}([x, x^*]) - [x, x^*] = 0$ *for all* $x \in \mathscr{R}$;
(*iii*) $\mathscr{R}$ *is commutative.*

Proof. Clearly, $(iii) \implies (i)$ and (ii) both. Then we need to prove that $(i) \implies (iii)$ and $(ii) \implies (iii)$.
To prove $(i) \implies (iii)$. Suppose that $\mathscr{R}$ satisfies $\mathfrak{F}([x, , x^*]) + [x, x^*] = 0$ for all $x \in \mathscr{R}$. If $\mathfrak{F} = 0$, then $[x, x^*] = 0$ for all $x \in \mathscr{R}$. Thus, Fact 2.3 yields the required result. Henceforward, we assume that $\mathfrak{F} \neq 0$ and we consider the case $\mathfrak{F}([x, , x^*]) + [x, x^*] = 0$ for all $x \in \mathscr{R}$. This can be rewritten as $\mathfrak{F}([x, , x^*]) + I_{\mathscr{R}}([x, x^*]) = 0$ (where $I_{\mathscr{R}}$ is the identity map on $\mathscr{R}$) for all $x \in \mathscr{R}$ and hence we obtain $(\mathfrak{F} + \mathfrak{I}_{\mathscr{R}})([x, , x^*]) = 0$ for all $x \in \mathscr{R}$. In view of Fact 4 of [1], we set $\mathfrak{F} + \mathfrak{I}_{\mathscr{R}} = \mathfrak{G}$, Then, the last relation reduces to $\mathfrak{G}([x, , x^*]) = 0$ for all $x \in \mathscr{R}$. Application of Theorem 3.1 gives the required conclusion.
By the same argument, we prove $\mathscr{R}$ is commutative in the case $\mathfrak{F}([x, x^*]) - [x, x^*] = 0$ for all $x \in \mathscr{R}$. Hence, $(i) \implies (iii)$ and $(ii) \implies (iii)$. Thereby theorem is proved. □

Using the similar arguments with necessary variations, one can prove the following.

Theorem 3.4. *Let $\mathscr{R}$ be a prime ring with involution$'*'$ of the second kind such that char$(\mathscr{R}) \neq 2$. Let $\mathfrak{F}$ be a generalized derivation of $\mathscr{R}$. Then the following conditions are mutually equivalent:*

(*i*) $\mathfrak{F}(x \circ x^*) + (x \circ x^*) = 0$ *for all* $x \in \mathscr{R}$;

(*ii*) $\mathfrak{F}(x \circ x^*) - (x \circ x^*) = 0$ *for all* $x \in \mathscr{R}$;

(*iii*) $\mathscr{R}$ *is commutative.*

The following are immediate consequences of our main results.

Corollary 3.1. *Let* $\mathscr{R}$ *be a prime ring with involution* $'*'$ *of the second kind such that* $char(\mathscr{R}) \neq 2$. *Let* $\mathfrak{F}$ *be a generalized derivation of* $\mathscr{R}$. *Then the following conditions are mutually equivalent:*

(*i*) $\mathfrak{F}(xx^*) + xx^* = 0$ *for all* $x \in \mathscr{R}$;

(*ii*) $\mathfrak{F}(xx^*) - xx^* = 0$ *for all* $x \in \mathscr{R}$;

(*iii*) $\mathfrak{F}(x) + x^* = 0$ *for all* $x \in \mathscr{R}$;

(*iv*) $\mathfrak{F}(x) - x^* = 0$ *for all* $x \in \mathscr{R}$;

(*v*) $\mathscr{R}$ *is commutative.*

Proof. Clearly, $(v) \implies (i)$, $(v) \implies (ii)$, $(v) \implies (iii)$, $(v) \implies (iv)$ and $(v) \implies (iv)$. Then we need to prove that $(i) \implies (v)$, $(ii) \implies (v)$, $(iii) \implies (v)$, $(iv) \implies (v)$. To prove $(i) \implies (v)$. We assume that $\mathscr{R}$ satisfies $\mathfrak{F}(xx^*) + xx^* = 0$ for all $x \in \mathscr{R}$. If $\mathfrak{F} = 0$, then $xx^* = 0$ for all $x \in \mathscr{R}$. This implies that $[x, x^*] = 0$ for all $x \in \mathscr{R}$. Fact 2.1 yields the required result. Henceforward, we assume that $\mathfrak{F} \neq 0$ and we consider the case $\mathfrak{F}(xx^*) + xx^* = 0$ for all $x \in \mathscr{R}$. Interchanging the role of x and x^*, and combining the obtained relation, we get $\mathfrak{F}([x, , x^*]) + [x, x^*] = 0$ for all $x \in \mathscr{R}$. Result follows by Theorem 3.3(i). By the same same argument with necessary variations, we obtain the desire conclusion in the case $\mathfrak{F}(xx^*) - xx^* = 0$ for all $x \in \mathscr{R}$.

To prove $(iii) \implies (v)$ and $(iv) \implies (v)$. We consider the case $\mathfrak{F}(x) + x^* = 0$ for all $x \in \mathscr{R}$, or $\mathfrak{F}(x) - x^* = 0$ for all $x \in \mathscr{R}$. Substituting $[x, x^*]$ for x and using the fact that $[x, x^*]^* = [x, x^*]$ for all $x \in \mathscr{R}$, we conclude that $\mathfrak{F}([x, x^*]) + [x, x^*] = 0$ for all $x \in \mathscr{R}$, or $\mathfrak{F}([x, x^*]) - [x, x^*] = 0$ for all $x \in \mathscr{R}$. Application of Theorem 3.3 yields the required conclusion. Hence, $(i) \implies (v)$, $(ii) \implies (v)$, $(iii) \implies (v)$ and $(iv) \implies (v)$. This proves the corollary. □

Corollary 3.2. *Let* $\mathscr{R}$ *be a prime ring with involution* $'*'$ *of the second kind such that* $char(\mathscr{R}) \neq 2$. *Let* δ *be a nonzero derivation of* $\mathscr{R}$. *Then the following conditions are mutually equivalent:*

(*i*) $\delta([x, x^*]) \pm [x, x^*] = 0$ *for all* $x \in \mathscr{R}$;

(*ii*) $\delta(x \circ x^*) \pm (x \circ x^*) = 0$ *for all* $x \in \mathscr{R}$;

(*iii*) $\delta(x) \pm x^* = 0$ *for all* $x \in \mathscr{R}$;

(*iv*) *R is commutative.*

Corollary 3.3 ([13, Theorem 3.4]). *Let* $\mathscr{R}$ *be a prime ring with involution* $'*'$ *of the second kind such that* $char(\mathscr{R}) \neq 2$. *Let* δ *be a nonzero derivation of* $\mathscr{R}$ *such that* $\delta([x, x^*]) \pm [x, x^*] = (0)$ *for all* $x \in \mathscr{R}$. *Then* $\mathscr{R}$ *is commutative.*

Corollary 3.4 ([13, Theorem 3.5]). *Let* $\mathscr{R}$ *be a prime ring with involution* $'*'$ *of the second kind such that* $char(\mathscr{R}) \neq 2$. *Let* δ *be a nonzero derivation of* $\mathscr{R}$ *such that* $\delta(x \circ x^*) \pm (x \circ x^*) = (0)$ *for all* $x \in \mathscr{R}$. *Then* $\mathscr{R}$ *is commutative.*

Corollary 3.5 ([18, Theorem 2.1 & 2.2 $for\ I = \mathscr{R}$]). *Let* $\mathscr{R}$ *be a prime ring with involution* $'*'$ *of the second kind such that* $char(\mathscr{R}) \neq 2$. *If* $\mathscr{R}$ *admits a generalized derivation* $\mathfrak{F}$ *with an associated nonzero derivation* δ *such that* $\mathfrak{F}([x, y]) \pm [x, y] = 0$ *for all* $x, y \in \mathscr{R}$, *then* $\mathscr{R}$ *is commutative.*

Corollary 3.6 ([18, Theorem 2.3 & 2.4 $for\ I = \mathscr{R}$]). *Let* $\mathscr{R}$ *be a prime ring with involution* $'*'$ *of the second kind such that* $char(\mathscr{R}) \neq 2$. *If* $\mathscr{R}$ *admits a generalized derivation* $\mathfrak{F}$ *with an associated nonzero derivation* δ *such that* $\mathfrak{F}(x \circ y) \pm (x \circ y) = 0$ *for all* $x, y \in \mathscr{R}$, *then* $\mathscr{R}$ *is commutative.*

Corollary 3.7. *Let* $\mathscr{R}$ *be a prime ring with involution*$'*'$ *of the second kind such that* $char(\mathscr{R}) \neq 2$. *Let* $\mathfrak{F}$ *be a generalized derivation of* $\mathscr{R}$. *Then the following conditions are mutually equivalent:*

(*i*) $\mathfrak{F}([x, y]) \pm [x, y] = 0$ *for all* $x, y \in \mathscr{R}$;

(*ii*) $\mathfrak{F}(x \circ y) \pm (x \circ y) = 0$ *for all* $x, y \in \mathscr{R}$;

(*iii*) $\mathfrak{F}(x) \pm x = 0$ *for all* $x \in \mathscr{R}$;

(*iii*) $\mathscr{R}$ *is commutative.*

It is worthwhile to mention here that in case if the associated derivation δ is zero, then the generalized derivations $\mathfrak{F}$ act as left centralizers(see [4] for further details). Hence, Theorems 3.3 & 3.4 of [4] becomes corollaries of our main results.

Remark 3.1. *At the end, let us also point out that we do not know yet whether these results are true for semiprime rings with involution* $'*'$ *of the second kind and with suitable torsion restrictions involving generalized derivations(or derivations). Hence, it is an open problem.*

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