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Forbidden Structures in Heyting Algebras with Respect to Sublattices

Manish Agalave

Department of Mathematics
Savitribai Phule Pune University
Pune 411 007, India

R. S. Shewale

Department of Mathematics
Savitribai Phule Pune University
Pune 411 007, India

Vilas Kharat

Department of Mathematics
Savitribai Phule Pune University
Pune 411 007, India

Abstract

In this paper, we have obtained forbidden structures of varieties of Heyting algebras namely $H_2, H_3, H_4, H_5, H_6, H_7$ with respect to sublattices.

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Keywords: Brouwerian lattice, Heyting Algebra, Distributive lattice, Sublattice, Forbidden stucture

1 Introduction

The study of forbidden structures has a long history. In fact, it has been noted and studied in depth by many researchers; see [1], [3], [4], [5], [6], [13], [17], [19], [20], [21], [22], [23], etc. A *Boolean lattice* is a complemented distributive lattice. It is well known that complements (if exist) are unique in any distributive lattice. It follows that any Boolean lattice is dually isomorphic with itself (self-dual); see [1]. A Boolean lattice can also be regarded as an algebra with two binary operations $\wedge$, $\vee$, an unary operation $'$ (complementation) and two nullary operations of picking up special elements namely 0 and 1. Thus a Boolean algebra is an algebra of the type $< B; \wedge, \vee, ', 0, 1 >$; see [6]. Boolean lattices considered as algebras (2, 2, 1, 0, 0) are called Boolean algebras. In a Boolean algebra A, for an element $a \in A$, there is a complement of a, denoted by $a' \in A$ which is the largest element $x \in A$ such that $a \wedge x = 0$. More generally, for $a, b, x \in A$, $a \wedge x \leq b$ if and only if $a \wedge x \wedge b' = 0$, i.e., $x \wedge (a \wedge b') = 0$ or $x \leq (a \wedge b')' = b \vee a'$. Therefore, given $a, b \in A$, there exists a largest element $c \in A$, $c = b \vee a'$ such that $a \wedge c \leq b$. Brouwer and Heyting characterized an important generalization of Boolean algebras through an extension of the preceding property as stipulated in the definition given below.

Definition 1.1 *A Brouwerian lattice L is a lattice in which, for any elements a and b, the set of all $x \in L$ such that $a \wedge x \leq b$ contains a greatest element. Such greatest element is called relative pseudocomplement of a in b, denoted as a^*_b or $a \rightarrow b$ or $b : a$, and so the lattice is also called relatively pseudocomplemented lattice. The operation of getting relative pseudocomplements is called Heyting operation.*

From the definition, it is easy to observe that every Brouwerian lattice has the unit element. In a Brouwerian lattice L with 0, the relative pseudocomplement of a in 0 is nothing but the pseudocomplement a^* of a. The study of different types of algebras can be seen in [7], [8], [9], [10] and [2]. The following result is proved in [1].

Theorem 1.2 ([1]) *Any Brouwerian lattice is distributive.*

According to [7], a Brouwerian lattice which is bounded, is called a *Heyting algebra*. The class of Heyting algebras is a subclass of the class of distributive lattices and also the class of pseudocomplemented lattices. A Heyting algebra H is said to be of order 3 if every interval in $D(H)$ is complemented, where $D(H) = \{x \in H : x^* = 0\}$. Heyting algebras are well studied in the literature. Especially, Heyting algebras have been found useful to analyze qualitative relations in biological systems and different biological processes (see [14], [15],

[16]). Qualitative relationships concern with those parts of biological systems that determine functional properties, and also different relations among them.

We have the following definitions.

A Heyting algebra is of order 3 (H_3) if and only if it satisfies the identity $x \vee x^*_{y \vee y^*} = 1$. (see [7])

A Heyting algebra is of order 2 (H_2) if and only if it satisfies the identity $x^* \vee x^{**} = 1$. This is also called as Stone algebra.

A Heyting algebra is of order 4 (H_4) if and only if it satisfies the identities $x \vee x^*_{y \vee y^*} = 1$ and $x^*_y \vee y^*_x \vee [x^*_{y^*} \wedge (y^*)^*_x] = 1$.

A Heyting algebra is of order 5 (H_5) if and only if it satisfies the identities $x \vee x^*_{y \vee y^*} = 1$ and $x^* \vee x^{**} = 1$.

A Heyting algebra is of order 6 (H_6) if and only if it satisfies the identity $x^*_y \vee y^*_x = 1$.

A Heyting algebra is of order 7 (H_7) if and only if it satisfies the identity $x \vee x^* = 1$. This is also called as Boolean algebra.

These varieties satisfy following inclusion:

$$H_7 \subset H_5 \subset H_6 \subset H_2,$$

and

$$H_7 \subset H_5 \subset H_4 \subset H_3.$$

2 Main Results

In this paper, all lattices considered are finite.

Lemma 2.1 *Let L be a Brouwerian lattice with 0 and 1. If $a, b \in L$ such that b is meet irreducible with $b < a$, then $a^*_b = b$.*

Proof. Suppose that L is a Brouwerian lattice with 0 and 1 and $a, b \in L$ are such that b is meet irreducible with $b < a$, with $a^*_b \neq b$. Since $b \leq a^*_b$, we must have $b < a^*_b$. Suppose there exists an element z in L such that $b \prec z \leq a$. Since b is meet irreducible, we get that $z \leq a^*_b$, which implies that $z \wedge a \leq a \wedge a^*_b$, and so $z \wedge a \leq b$, i.e., $z \leq b$, a contradiction to the fact that $b \prec z$. We conclude that, whenever b is meet irreducible and $b < a$, we must have $b = a^*_b$. Hence $a^*_b = b$. □

The following results are given in [6].

Theorem 2.2 ([6]) *Identities are preserved under the formation of sublattices, homomorphic images, direct products, and ideal lattices.*

Distributivity criteria in terms of forbidden structure is given by Birkhoff and modularity criteria in terms of forbidden structure is given by Dedekind.

Theorem 2.3 ([6]) *A lattice L is distributive if and only if L does not contain a pentagon (N_5) or a diamond (M_3).*

Theorem 2.4 *A Heyting algebra L is in H_7 if and only if it does not contain a sublattice isomorphic to the lattice as depicted in the Figure-1.*

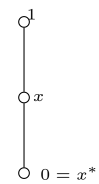

Figure 1:

Proof. If there exists $x \in L$ such that $x \neq 1$ and $x^* = 0$, then $x \vee x^* = x \vee 0 = x \neq 1$. Hence $L \notin H_7$ and the lattice as depicted in the Figure-1 is not in H_7.

Conversely, suppose $L \notin H_7$. Then there exists an element, say $x \in L$ such that $x \vee x^* \neq 1$. Suppose that $x^* = 0$, then the sublattice $\{0 = x^*, x = x \vee x^*, 1\}$ of L is isomorphic to the lattice depicted in the Figure-1. Now, suppose $x^* \neq 0$, then $0, x, x^*, x \vee x^*, 1$ are distinct elements of L. Take $y = x \vee x^*$, then the sublattice $\{0 = y^*, y = y \vee y^*, 1\}$ of L is isomorphic to the lattice depicted in the Figure-1. □

Theorem 2.5 *A Heyting algebra L is in H_2 if and only if it does not contain a sublattice containing 0 of L isomorphic to the lattice as depicted in the Figure-2.*

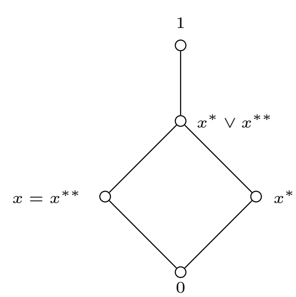

Figure 2:

Proof. If there exists $x \in L$ such that $x = x^{**}$ and $x^* \vee x^{**} \neq 1$, then $L \notin H_2$ and the lattice as depicted in the Figure-2 is not in H_2.

Conversely, suppose $L \notin H_2$. Then there exists an element, say $x \in L$ such that $x^* \vee x^{**} \neq 1$. Suppose $x = x^{**}$, then $\{0, x, x^*, x \vee x^*, 1\}$ forms a sublattice isomorphic to the lattice depicted in Figure-2. Now, suppose that $x \neq x^{**}$ But then, $x < x^{**}$ and $0, x, x^*, x^{**}, 1$ are distinct elements of L. Note that $x \vee x^* \neq x^* \vee x^{**}$; otherwise, $\{0, x, x^*, x^{**}, x \vee x^* = x^* \vee x^{**}, 1\}$ forms a sublattice isomorphic to N_5, contradicting distributivity of L. Also, we have $x^{**} \wedge (x \vee x^*) = (x^{**} \wedge x) \vee (x^{**} \wedge x^*) = x \vee 0 = x$. Take $y = x^*$, then the sublattice $\{0, y = y^{**}, y^*, y^* \vee y^{**}, 1\}$ of L is isomorphic to the lattice depicted in Figure-2. □

Theorem 2.6 *A Heyting algebra L is in H_6 if and only if it does not contain a sublattice isomorphic to the lattice as depicted in the Figure-2.*

Proof. If there exists $x, y \in L$ such that $x_y^* \vee y_x^* \neq 1$, then $L \notin H_6$ and the lattice as depicted in the Figure-2 is not in H_6.

Conversely, suppose $L \notin H_6$. Then there exists a pair of elements $x, y \in L$ such that $x_y^* \vee y_x^* \neq 1$. Note that $x \parallel y$; otherwise, if $x \nparallel y$, then either $x \leq y$ or $y \leq x$ and $x \leq y$ gives $x_y^* = 1$, a contradiction to the fact $x_y^* \vee y_x^* \neq 1$ and similarly for $y \leq x$. Also, we must have $x \nleq x_y^*$ and $y \nleq y_x^*$. Indeed, if $x \leq x_y^*$, then $x = x \wedge x \leq x \wedge x_y^* \leq y$, a contradiction to the fact that $x \parallel y$ and similarly for $y \nleq y_x^*$. Now, by definition, $y \leq x_y^*$ and $x \leq y_x^*$ and accordingly we have the following cases.

Case **1** Suppose $y = x_y^*$ and $x = y_x^*$. Then $\{x \wedge y, x, y, x \vee y, 1\}$ forms a sublattice isomorphic to the lattice depicted in Figure-2.

Case **2** Suppose $y = x_y^*$ and $x \neq y_x^*$. Then $x \wedge y, x, y, y_x^*, x \vee y, 1$ are distinct elements of L. Also, we have the following.

(**2-I**) We claim that $y_x^* \wedge (x \vee y) = x$. Indeed, if $y_x^* \wedge (x \vee y) = z \neq x$, then $x < z < x \vee y$ and the set $\{x \wedge y, x, y, z, x \vee y\}$ forms a sublattice of L isomorphic to N_5.

(**2-II**) We claim that $y_x^* \wedge y = x \wedge y$. If $y_x^* \wedge y = z$, then $x \wedge y \leq z$ since $x \leq y_x^*$. By definition we get, $y \wedge y_x^* \leq x$, i.e., $z \leq x$. Also, we have $z \leq y$ and so $z \leq x \wedge y$ and consequently $z = x \wedge y$.

(**2-III**) We claim that $x \vee y \neq (y_x^*) \vee y$. Indeed, if $x \vee y = (y_x^*) \vee y$, then $x \wedge y, x, y, y_x^*, x \vee y$ are distinct elements of L and form a sublattice of L that is isomorphic to N_5.

Now, the set $\{x \wedge y, x, y, x \vee y, y_x^*, y_x^* \vee y, 1\}$ of distinct elements of L. Take $u = y_x^*, v = x_y^*$, then the distinct elements $\{u \wedge v, u, v, u \vee v, 1\}$ of L forms a sublattice of L that is isomorphic to the lattice depicted in Figure-2.

Case **3** Similar to the Case **2**, if $y \neq x_y^*$ and $x = y_x^*$, we get a sublattice of L that is isomorphic to the lattice depicted in Figure-2.

Case **4** Suppose $y \neq x_y^*$ and $x \neq y_x^*$, then $y < x_y^*$ and $x < y_x^*$. Consider the set $\{x \wedge y, x, y, y_x^*, x_y^*, 1\}$ of distinct elements of L. Now, as in Case **2**, we have $y_x^* \wedge y = x \wedge y$ and $x_y^* \wedge x = x \wedge y$. Also, $x_y^* \vee (x \vee y) = (x_y^* \vee y) \vee x = x_y^* \vee x$. Similarly, $y_x^* \vee (x \vee y) = y_x^* \vee y$.
Next, note that $y_x^* \vee y \neq x \vee y$. Indeed, if $y_x^* \vee y = x \vee y$, then the set $\{x \wedge y, x, y, y_x^*, x \vee y\}$ of distinct elements of L is a sublattice isomorphic to N_5. Similarly $x_y^* \vee x \neq x \vee y$.
Also, $y_x^* \vee y \neq y_x^*$. Indeed, if $y_x^* \vee y = y_x^*$, then $y \leq y_x^*$, a contradiction to the fact that $y \nleq y_x^*$. Similarly $x_y^* \vee x \neq x_y^*$.
Now, we have $x_y^* \vee x \neq x_y^*, y_x^* \vee y \neq y_x^*, y_x^* \vee y \neq x \vee y, x_y^* \vee x \neq x \vee y$ and the following subcases.

(4-I) Suppose $x_y^* \vee x = y_x^* \vee y = y_x^* \vee x_y^*$ and $y_x^* \wedge x_y^* \neq x \wedge y$, then $x \wedge y, x, y, x \vee y, x_y^*, y_x^*, y_x^* \vee x_y^*, y_x^* \wedge x_y^*, 1$ are distinct elements of L. Take $u = y_x^*, v = x_y^*$, then the distinct elements $\{u \wedge v, u, v, u \vee v, 1\}$ of L forms a sublattice of L that is isomorphic to the lattice depicted in Figure-2.

(4-II) Suppose $x_y^* \vee x \neq y_x^* \vee y = y_x^* \vee x_y^*$ and $y_x^* \wedge x_y^* \neq x \wedge y$. Then $y_x^*, x \vee y, y_x^* \wedge (x \vee x_y^*), x \vee x_y^*, y_x^* \vee x_y^*$ are distinct elements of L and form a sublattice of L that is isomorphic to N_5.

(4-III) Suppose $x_y^* \vee x \neq y_x^* \vee y \neq y_x^* \vee x_y^*$ and $y_x^* \wedge x_y^* \neq x \wedge y$. We claim that $x \wedge x_y^* = x \wedge y$. Indeed, if $x \wedge x_y^* = z \neq x \wedge y$, then $x \wedge y, z, y_x^* \vee x_y^*, y$ are distinct elements of L and form a sublattice of L that is isomorphic to N_5. Next, consider $x_y^* \wedge (x \vee y) = (x_y^* \wedge x) \vee (x_y^* \wedge y) = (x \wedge y) \vee y = y$ and similarly $y_x^* \wedge (x \vee y) = x$ which implies that $y_x^* \wedge x_y^* = x \wedge y$, a contradiction to the fact that $y_x^* \wedge x_y^* \neq x \wedge y$, so this case will not arise.

(4-IV) Suppose $x_y^* \vee x = y_x^* \vee y = y_x^* \vee x_y^*$ and $y_x^* \wedge x_y^* = x \wedge y$. Then $\{x \wedge y, x_y^*, y_x^*, y, y_x^* \vee x_y^*\}$ is a set of distinct elements and is a sublattice of L that is isomorphic to N_5.

(4-V) Suppose $x_y^* \vee x \neq y_x^* \vee y = y_x^* \vee x_y^*$ and $y_x^* \wedge x_y^* = x \wedge y$. Then $\{x \wedge y, x_y^*, y_x^*, y, y_x^* \vee x_y^*\}$ is a set of distinct elements and is a sublattice of L that is isomorphic to N_5.

(4-VI) Suppose $x_y^* \vee x \neq y_x^* \vee y \neq y_x^* \vee x_y^*$ and $y_x^* \wedge x_y^* = x \wedge y$. Then $\{x \wedge y, x, y, x \vee y, x_y^*, y_x^*, y_x^* \vee x_y^*, x_y^* \vee x, y_x^* \vee y, 1\}$ is a set of distinct elements of L. Take $u = y_x^*, v = x_y^*$, then the distinct elements $\{u \wedge v, u, v, u \vee v, 1\}$ of L forms a sublattice of L that is isomorphic to the lattice depicted in Figure-2. □

Theorem 2.7 *A Heyting algebra L is in H_3 if and only if it does not contain a sublattice isomorphic to the lattice as depicted in the Figure-3.*

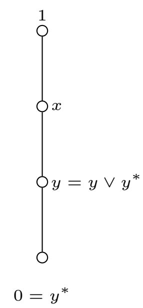

Figure 3:

Proof. Suppose there exists a four element chain, $0 < y < x < 1$ with $y^* = 0$ and $x_y^* = y$. Consequently, $y^* \vee y = 0 \vee y = y$ and $x \vee x^*_{y \vee y^*} = x \vee x^*_y = x \vee y = x \neq 1$ and so $L \notin H_3$, and the lattice as depicted in the Figure-3 is not in H_3.

Conversely, suppose that $L \notin H_3$. Then there exists a pair of elements $x, y \in L$ such that $x \vee x^*_{y \vee y^*} \neq 1$. Now, we have $x \neq 1, x \neq 0, x \nleq y, y \neq 0, y^* \neq 1, x \nleq y^*, x \nleq y \vee y^*, x \neq y \vee y^*, x \neq y, x \neq y^*$. If either of these conditions is violated, then we get $x \vee x^*_{y \vee y^*} = 1$. If $y^* = 0$ and $y < x$, then $y = y \vee y^* = x^*_{y \vee y^*}$ and so $\{x, y^* = 0, y = y \vee y^*, 1\}$ is a sublattice of L isomorphic to the lattice depicted in Figure-3. Suppose that $y^* \neq 0$ and $y \not< x$. Therefore $0, y, x, y^*, y \vee y^*, 1$ are distinct elements of L, and accordingly we have the following cases.

Case **1** Suppose $y \vee y^* < x$. Then $\{0, y, x, y^*, y \vee y^*, 1\}$ are distinct elements of L. Take $z = x^*_{y \vee y^*}$, then $\{0, z, x, 1\}$ is a sublattice of L isomorphic to the lattice depicted in Figure-3.

Case **2** Suppose $(y \vee y^*) \parallel x$. We have the following subcases.

(**2-I**) Suppose $y < x$ and $y^* \parallel x$.

(**2-I-i**) If $y = x \wedge (y \vee y^*)$ and $x \wedge y^* = 0$, then $\{0, y, x, y^*, y \vee y^*, x \vee y^*, 1\}$ are distinct elements of L. Take $z = y \vee y^*$, then $\{0, z, x, 1\}$ is a sublattice of L isomorphic to the lattice depicted in Figure-3.

(**2-I-ii**) If $y \neq x \wedge (y \vee y^*)$ and $x \wedge y^* \neq 0$, then $\{0, y, x, y^*, y \vee y^*, x \vee y^*, x \wedge y^*, x \wedge (y \vee y^*), 1\}$ are distinct elements of L. Take $z = y \vee y^*, u = x \vee y^*$, then $\{0, z, u, 1\}$ is a sublattice of L isomorphic to the lattice depicted in Figure-3.

(2-I-iii) If $y \neq x \wedge (y \vee y^*)$ and $x \wedge y^* = 0$, then $x \wedge (y \vee y^*) = (x \wedge y) \vee (x \wedge y^*) = y \vee 0 = y$, which implies that $y = x \wedge (y \vee y^*)$ and this case will not arise.

(2-I-iv) If $y = x \wedge (y \vee y^*)$ and $x \wedge y^* \neq 0$, then $y = x \wedge (y \vee y^*) = (x \wedge y) \vee (x \wedge y^*) = y \vee (x \wedge y^*)$, which implies that $x \wedge y^* \leq y$ and $x \wedge y^* = 0$ and this case will not arise.

(2-II) Suppose $y^* < x$ and $y \parallel x$. But then $x \wedge y \neq 0$.

(2-II-i) If $y^* \neq x \wedge (y \vee y^*)$, then $\{0, y, x, y^*, y \vee y^*, x \wedge y, x \wedge (y \vee y^*), y \vee x, 1\}$ are distinct elements of L. Take $z = y \vee y^*, u = x \vee y$, then $\{0, z, u, 1\}$ is a sublattice of L isomorphic to the lattice depicted in Figure-3.

(2-II-ii) If $y^* = x \wedge (y \vee y^*)$, then $y^* = x \wedge (y \vee y^*) = (x \wedge y) \vee (x \wedge y^*) = y^* \vee (x \wedge y)$, which implies that $x \wedge y \leq y^*$ and so $x \wedge y = 0$ and this case will not arise.

(2-III) Suppose $y^* \parallel x$ and $y \parallel x$. But then, $x \wedge y \neq 0$ and $x \parallel (y \vee y^*)$. Note that $y^* \vee (x \wedge y) < y \vee y^*$. Indeed, if $y^* \vee (x \wedge y) = y \vee y^*$, then the set of distinct elements, namely $\{0, y, y^*, x \wedge y, y \vee y^*\}$ forms a sublattice of L isomorphic to N_5. Now, $0, y, x, y^*, x \wedge y, y \vee y^*, y^* \vee (x \wedge y), 1$ are distinct elements of L. We claim that $x \vee y, x \vee y^*, y \vee y^*, x \vee (y \vee y^*)$ are mutually distinct elements of L. If $x \vee y = x \vee y^* = y \vee y^* = x \vee (y \vee y^*)$, then $\{x, y, y^* \vee (x \wedge y), x \wedge y, x \vee (y \vee y^*)\}$ is a sublattice of L isomorphic to M_3. If $x \vee y = x \vee y^*$, then $(x \vee y) \vee y^* = (x \vee y^*) \vee y^* = x \vee y^*$, which is not possible. If $x \vee y = y \vee y^*$ or $x \vee y^* = y \vee y^*$, then $x \leq y \vee y^*$, a contradiction to the assumption. Thus $\{0, y, y^*, x, x \wedge y, y \vee y^*, y^* \vee (x \wedge y), x \vee y, x \vee y^*, x \vee (y \vee y^*), 1\}$ is a set of distinct elements of L. Now we have the following two subcases.

(2-III-i) Suppose $x \wedge [y^* \vee (x \wedge y)] = x \wedge y$. Note that $x \wedge y = x \wedge [y^* \vee (x \wedge y)] = (x \wedge y^*) \vee [x \wedge (x \wedge y)] = (x \wedge y^*) \vee (x \wedge y)$ which implies that $x \wedge y^* \leq x \wedge y$ and so $x \wedge y^* = 0$. Also, $y \vee (x \wedge y^*) = y \vee 0 = y$. Hence, $\{0, y, y^*, x, x \wedge y, y \vee y^*, y^* \vee (x \wedge y), x \vee y, x \vee y^*, x \vee (y \vee y^*), 1\}$ is a set of distinct elements of L. Take $z = y \vee y^*, u = x \vee (y \vee y^*)$, then $\{0, z, u, 1\}$ is a sublattice of L isomorphic to the lattice depicted in Figure-3.

(2-III-ii) Suppose $x \wedge [y^* \vee (x \wedge y)] \neq x \wedge y$.
If $x \wedge [y^* \vee (x \wedge y)] \neq x \wedge y$, then $x \wedge y < x \wedge [y^* \vee (x \wedge y)]$. Note that $x \wedge y^* \neq 0$. Indeed, if $x \wedge y^* = 0$, then $x \wedge [y^* \vee (x \wedge y)] = (x \wedge y^*) \vee [x \wedge (x \wedge y)] = (x \wedge y^*) \vee (x \wedge y) = 0 \vee (x \wedge y) = x \wedge y$, a contradiction to the assumption. Therefore $\{0, y, y^*, x, x \wedge y, x \wedge (y \vee y^*), y \vee y^*, y^* \vee (x \wedge y), x \vee y, x \vee y^*, x \vee (y \vee y^*), x \wedge [y^* \vee (x \wedge y)], x \wedge y^*, 1\}$ is a set of distinct elements of L. Take $z = y \vee y^*, u = x \vee (y \vee y^*)$,

then $\{0, z, u, 1\}$ is a sublattice of L isomorphic to the lattice depicted in Figure-3. □

Proof of the following Theorem follows from the Theorems-2.5 and 2.7.

Theorem 2.8 *A Heyting algebra L is in H_5 if and only if it does not contain a sublattice isomorphic to one of the lattices as depicted in the Figure-2 or 3.*

Theorem 2.9 *A Heyting algebra L is in H_4 if and only if it does not contain a sublattice isomorphic to one of the lattices as depicted in the Figure-3 or 4.*

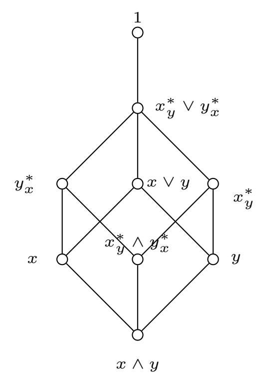

Figure 4:

Proof. Suppose there exists a four element chain, $0 < y < x < 1$ with $y^* = 0$ and $x_y^* = y$. Consequently, $y^* \vee y = 0 \vee y = y$ and $x \vee x_{y \vee y^*}^* = x \vee x_y^* = x \vee y = x \neq 1$ and so $L \notin H_3$, and the lattice as depicted in the Figure-3 is not in H_4. If for $x, y \in L$ such that $x_y^* \vee y_x^* \vee [x_{y^*}^* \wedge (y^*)_x^*] \neq 1$, then L is not in H_4 and the lattice as depicted in the Figure-4 is not in H_4.

Conversely, suppose $L \notin H_4$. Then it is a either not in H_3 or there exists a pair of elements $x, y \in L$ such that $x_y^* \vee y_x^* \vee (x_{y^*}^* \wedge (y^*)_x^*) \neq 1$. Suppose it is not in H_3, then by Theorem-2.7, L contains a sublattice isomorphic to the lattice depicted in Figure-3. Now, suppose it is in H_3, then we have a pair of elements $x, y \in L$ such that $x_y^* \vee y_x^* \vee (x_{y^*}^* \wedge (y^*)_x^*) \neq 1$. Now, we have $x \nleq y, y \nleq x, x \neq y^*, y \neq 0, y \neq 1, x \neq 0, x \neq 1$. If either of these conditions is violated, then $x_y^* \vee y_x^* \vee (x_{y^*}^* \wedge (y^*)_x^*) = 1$. Also, we have $x \parallel y, x \parallel x_y^*, y \parallel y_x^*, x \parallel x_{y^*}^*, y^* \parallel (y^*)_x^*, y \leq x_y^*, x \leq y_x^*, y^* \leq x_{y^*}^*, x \leq (y^*)_x^*$. Therefore, $0, y, x, y^*, y \vee y^*, 1$ are distinct elements of L.

Case **1** Suppose $x < y^*$ and so $x \wedge y = 0, y^* \leq y_x^*$. Also, $x < y^*$ and so $x_{y^*}^* = 1 \neq y^*$.

(**1-I**) Suppose $y^* = y_x^*$. We have the following subcases.

(**1-I-i**) For $x \vee y = (y^*)^*_x$.
(**1-I-i-**A) If $y = x^*_y$, then $\{0, y, y^*, x, x \vee y, y \vee y^*, 1\}$ are distinct elements of L. Take $u = x \vee y, v = y \vee y^*$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.
(**1-I-i-**B) If $y \neq x^*_y$, we claim that $x \vee y \parallel x^*_y$. Indeed, if $x \vee y \leq x^*_y$, then $x = x \wedge (x \vee y) \leq x \wedge (x^*_y) \leq y$, a contradiction to the fact $x \parallel y$. Now, if $x^*_y < x \vee y$, then $\{0, x, y, y \vee x, x^*_y\}$ is a set of distinct elements of L and is a sublattice of L isomorphic to N_5.
(**1-I-i-**B-a) If $x^*_y \wedge y^*_x = 0$, then $\{0, x, y, y \vee x, x^*_y, y^*_x\}$ is a set of distinct elements of L. If $x \vee x^*_y < x^*_y \vee y^*_x$, then $y^* \wedge (x \vee x^*_y) = x$ with $(y^*)^*_x < x \vee x^*_y$ which is not possible and so $x \vee x^*_y = x^*_y \vee y^*_x$. Now, if $y \vee y^*_x < x^*_y \vee y^*_x$, then the set of distinct elements, namely $\{y, x^*_y, x^*_y \vee y^*_x, x \vee y, y \vee y^*_x\}$ is a sublattice of L isomorphic to N_5 and so $x \vee x^*_y = y \vee y^*_x = x^*_y \vee y^*_x$, then the set of distinct elements, namely $\{0, y^*, x^*_y, y, y \vee y^*\}$ is a sublattice of L isomorphic to N_5. Hence this case will not arise.
(**1-I-i-**B-b) If $x^*_y \wedge y^*_x \neq 0$, then similar to previous Case (**1-I-i-**B-a) we have $x \vee x^*_y = y \vee y^*_x = x^*_y \vee y^*_x$, then the sublattice $\{0, y, y^* = y^*_x, x, x \vee y, x^*_y, x^*_y \vee y^*_x = y \vee y^*, 1\}$ of L is isomorphic to the lattice depicted in Figure-4.

(**1-I-ii**) If $x \vee y \neq (y^*)^*_x$, then $(x \vee y) \wedge y^* = (x \wedge y^*) \vee (y \wedge y^*) = x \vee 0 = x$ which implies $x \vee y \leq (y^*)^*_x$ and so $x \vee y < (y^*)^*_x$.
(**1-I-ii-**A) If $y = x^*_y$, then $(x \vee y) \wedge y^* = x$. We claim that $(y^*)^*_x \wedge (y \vee y^*) = x \vee y$. Indeed, if $z = (y^*)^*_x \wedge (y \vee y^*) > x \vee y$, then the set of distinct elements, namely $\{x, y^*, y \vee x, z, y \vee y^*\}$ is a sublattice of L isomorphic to N_5. We claim that $(y^*)^*_x \wedge y^* = x$. Indeed, if $z = (y^*)^*_x \wedge y^* > x$, then the set of distinct elements, namely $\{0, x, y, y \vee x, z\}$ is a sublattice of L isomorphic to N_5. Therefore $\{0, y, y^*, x, x \vee y, y \vee y^*, (y^*)^*_x, (y^*)^*_x \vee y^*_x, 1\}$are distinct elements of L. Take $u = x \vee y, v = (y^*)^*_x$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.
(**1-I-ii-**B) For $y \neq x^*_y$.
(**1-I-ii-**B-a) If $(y^*)^*_x = (x \vee y) \vee x^*_y$, then $\{0, y, y^*, x, x \vee y, y \vee y^*, (y^*)^*_x, (y^*)^*_x \vee y^*_x, x^*_y, 1\}$ are distinct elements of L. Take $u = x \vee y, v = (y^*)^*_x \vee y^*_x$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.
(**1-I-ii-**B-b) If $(y^*)^*_x \neq (x \vee y) \vee x^*_y$, then $\{0, y, y^*, x, x \vee y, y \vee y^*, (y^*)^*_x, x \vee x^*_y, (y^*)^*_x \vee y^*_x, x^*_y, x^*_y \vee y^*_x, 1\}$ are distinct elements

of L. Take $u = x \vee y, v = (y^*)^*_x \vee y^*$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

(**1-II**) If $y^* \neq y^*_x$, then $y^* < y^*_x$ and so $0 \leq y \wedge y^*_x \leq x$. If $y \wedge y^*_x = x$, then $x \leq y$, a contradiction to the fact $x \parallel y$ and so $y \wedge y^*_x \neq x$. Now, if $y \wedge y^*_x < x$, then $y \wedge (y \wedge y^*_x) \leq y \wedge x = 0$, which implies that $y \wedge (y \wedge y^*_x) = (y \wedge y) \wedge y^*_x = y \wedge y^*_x = 0$ and so $y^*_x \leq y^*$, a contradiction to the fact $y^* < y^*_x$. Hence this case will not arise.

Case **2** Suppose $y^* < x$ and consequently $x \wedge y \neq 0$ and $(y^*)^*_x = 1 \neq x$. Also, $y^* < x \leq y^*_x$ and $x \vee y \neq y^*_x$.

(**2-I**) For $y = x^*_y$, we have the following subcases.

(**2-I-i**) If $x \neq y^*_x$, then $\{0, y, y^*, x, x \vee y, y^*_x, y \vee y^*_x, x \wedge y, 1\}$ are distinct elements of L. Take $u = x, v = y \vee y^*_x$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

(**2-I-ii**) If $x = y^*_x$, then $\{0, y, y^*, x, x \vee y, x \wedge y, 1\}$ are distinct elements of L. Take $u = x, v = y \vee y^*$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

(**2-II**) Suppose $y < x^*_y$. We have the following subcases.

(**2-II-i**) If $x \neq y^*_x$, then $\{0, y, y^*, x, x \vee y, y^*_x, x^*_y, x \vee x^*_y, x^*_y \vee y^*_x, y \vee y^*_x, x \wedge y, 1\}$ are distinct elements of L. Take $u = x, v = y^*_x$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

(**2-II-ii**) If $x = y^*_x$, then $\{0, y, y^*, x, x \vee y, x^*_y, x \vee x^*_y, x \wedge y, 1\}$ are distinct elements of L. Take $u = x, v = x \vee y$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

Case **3** Suppose $x \parallel y, x \parallel y^*$ and so $x \wedge y \neq 0, y^* \leq y^*_x$ and $y \leq (y^*)^*_x$. We claim that $y^* < y^*_x$. Indeed, if $y^* = y^*_x$, then $x \leq y^*_x = y^*$, a contradiction to the fact $x \parallel y^*$. Similarly, $y = (y^*)^*_x$. Therefore, $0, 1, x, y, y^*, x \wedge y, y^*_x, (y^*)^*_x$ are distinct elements of L.

(**3-I**) Suppose $y \vee y^* = y^*_x \vee (y^*)^*_x$. We have the following subcases.

(**3-I-i**) If $x \wedge y^* \neq 0$, then the sublattice $\{0, 1, x, y, y^*, x \wedge y, x \wedge y^*, y^*_x, (y^*)^*_x, y \vee y^*\}$ are distinct elements of L. Take $u = x, v = y \vee y^*$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

(3-I-ii) If $x \wedge y^* = 0$, then the set of distinct elements, namely $\{0, x, y^*, x \wedge y, y^*_x\}$ is a sublattice of L isomorphic to N_5. So this case will not arise.

(3-II) Suppose $y \vee y^* \neq y^*_x \vee (y^*)^*_x$. We have the following subcases.

(3-II-i) If $x \wedge y^* \neq 0$, then $\{0, 1, x, y, y^*, x \wedge y, x \wedge y^*, y^*_x, (y^*)^*_x, y \vee y^*, y^*_x \vee (y^*)^*_x, (x \wedge y) \vee y^*, (x \wedge y^*) \vee y, x \wedge (y \vee y^*)\}$ are distinct elements of L. Take $u = x, v = (y^*)^*_x \vee y^*$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3.

(3-II-ii) If $x \wedge y^* = 0$, then $\{0, 1, x, y, y^*, x \wedge y, x \wedge y^*, y^*_x, (y^*)^*_x, y \vee y^*, y^*_x \vee (y^*)^*_x\}$ are distinct elements of L. Take $u = y \vee y^*, v = x*_y$, then $\{0, u, v, 1\}$ forms a sublattice of L isomorphic to the lattice depicted in Figure-3, a contradiction to the fact that L is in H_3. □

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