International Journal of Algebra, Vol. 10, 2016, no. 1, 33 - 40
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ija.2016.51278

Direct Product of B-algebras

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Abstract

In this paper, we introduce the direct product of B-algebras and we obtain some of its properties.

Mathematics Subject Classification: 06F35

Keywords: Direct product of B-algebras, subalgebra, B-homomorphism, normality

1 Introduction

In 2002, the concept of B-algebras [6] was introduced by J. Neggers and H.S. Kim. A *B-algebra* $\mathbf{A} = (A; *, 0)$ is an algebra of type $(2, 0)$, that is, a nonempty set A together with a binary operation $*$ and a constant 0 satisfying the following axioms for all $x, y, z \in A$: (I) $x * x = 0$, (II) $x * 0 = x$, and (III) $(x * y) * z = x * (z * (0 * y))$. In the same paper, the concept of commutative B-algebras was also introduced. A B-algebra $\mathbf{A}$ is *commutative* if $x * (0 * y) = y * (0 * x)$ for all $x, y \in A$. H.S. Kim and H.G. Park [4] characterized commutativity of B-algebras. In [7], J. Neggers and H.S. Kim introduced the notions of subalgebras and normality in B-algebras, and established their properties. A nonempty subset N of A is called a *subalgebra* of $\mathbf{A}$ if $x * y \in N$

for any $x, y \in N$. By (I), 0 is always an element of a subalgebra. A nonempty subset N of A is called a *normal subalgebra* of $\mathbf{A}$ if $(x * a) * (y * b) \in N$ for any $x * y, a * b \in N$. A. Walendziak [9] characterized normality in B-algebras. J. Neggers and H.S. Kim used the concept of normality in B-algebras to construct quotient B-algebras. That is, given a normal subalgebra N of a B-algebra $\mathbf{A}$, the relation $\sim_N$ is defined by $x \sim_N y$ if and only if $x * y \in N$ for any $x, y \in A$. Then $\sim_N$ is a congruence relation of $\mathbf{A}$. For $x \in A$, we write xN for the congruence class containing x, that is, $xN = \{y \in A : x \sim_N y\}$. Denote $A/N = \{xN : x \in A\}$ and define $*'$ on A/N by $xN *' yN = (x * y)N$. Note that $xN = yN$ if and only if $x \sim_N y$. The algebra $\mathbf{A}/N = (A/N; *', N)$ is a B-algebra, and is called the *quotient B-algebra* of $\mathbf{A}$ modulo N. The concept of B-homomorphism was also introduced by J. Neggers and H.S. Kim. A map $\varphi : A \to B$ is called a *B-homomorphism* if $\varphi(x * y) = \varphi(x) * \varphi(y)$ for any $x, y \in A$. The *kernel of* φ, denoted by ker φ, is defined to be the set $\{x \in A : \varphi(x) = 0_B\}$. The ker φ is a normal subalgebra of $\mathbf{A}$, and ker $\varphi = \{0_A\}$ if and only if φ is one-one. A B-homomorphism φ is called a *B-monomorphism*, *B-epimorphism*, or *B-isomorphism* if φ is one-one, onto, or a bijection, respectively. In [7], the first and third isomorphism theorems for B-algebras are established. In [3], J.C. Endam and J.P. Vilela established the second isomorphism theorem for B-algebras. In this paper, we introduced the direct product of B-algebras and established some of its properties.

2 Direct Product of B-algebras

We begin with some examples of B-algebras.

Example 2.1 The algebra $(\mathbb{Z}; *, 0)$ is a B-algebra, where $*$ is defined by $x * y = x - y$ for all $x, y \in \mathbb{Z}$.

Example 2.2 [6] Let $A = \{0, 1, 2, 3, 4, 5\}$ be a set with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(A; *, 0)$ is a B-algebra.

Let $\mathbf{A} = (A; *, 0_A)$ and $\mathbf{B} = (B; *, 0_B)$ be B-algebras. Define the direct product of $\mathbf{A}$ and $\mathbf{B}$ to be the structure $\mathbf{A} \times \mathbf{B} = (A \times B; \circledast, (0_A, 0_B))$, where

$A \times B$ is the set $\{(a,b) : a \in A \text{ and } b \in B\}$ and whose binary operation $\circledast$ is given by $(a_1, b_1) \circledast (a_2, b_2) = (a_1 * a_2, b_1 * b_2)$. Note that the binary operation $\circledast$ is componentwise. Thus, the properties (I), (II), and (III) of $\mathbf{A} \times \mathbf{B}$ follow from those of $\mathbf{A}$ and $\mathbf{B}$. Hence, the following theorem easily follows.

Theorem 2.3 *The direct product of two B-algebras is also a B-algebra.*

Now, we extend this direct product to any finite family of B-algebras and obtain some of its properties. Let $\mathrm{I}_n = \{1, 2, \ldots, n\}$ and let $\{\mathbf{A}_i = (A_i; *, 0_i)$: $i \in \mathrm{I}_n\}$ be a finite family of B-algebras. Define the direct product of B-algebras $\mathbf{A}_1, \ldots, \mathbf{A}_n$ to be the structure $\prod_{i=1}^{n} \mathbf{A}_i = \left(\prod_{i=1}^{n} A_i; \circledast, (0_1, \ldots, 0_n) \right)$, where

$$\prod_{i=1}^{n} A_i = A_1 \times \cdots \times A_n = \{(a_1, \ldots, a_n) : a_i \in A_i, i \in \mathrm{I}_n\}$$

and whose operation $\circledast$ is given by

$$(a_1, \ldots, a_n) \circledast (b_1, \ldots, b_n) = (a_1 * b_1, \ldots, a_n * b_n).$$

Obviously, $\circledast$ is a binary operation on $\prod_{i=1}^{n} \mathbf{A}_i$.

Corollary 2.4 *If* $\{\boldsymbol{A}_i = (A_i; *, 0_i)$: $i \in I_n\}$ *is a family of B-algebras, then* $\prod_{i=1}^{n} \boldsymbol{A}_i$ *is a B-algebra.*

Theorem 2.5 *Let* $\{\boldsymbol{A}_i = (A_i; *, 0_i)$: $i \in I_n\}$ *be a family of B-algebras. Then each* $\boldsymbol{A}_i$ *is commutative if and only if* $\prod_{i=1}^{n} \boldsymbol{A}_i$ *is commutative.*

Proof: Let each $\mathbf{A}_i$ be commutative. If $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \prod_{i=1}^{n} A_i$, then $a_i, b_i \in A_i$ and $a_i * (0_i * b_i) = b_i * (0_i * a_i)$ for all $i \in \mathrm{I}_n$. Thus,

$$\begin{aligned} (a_1, \ldots, a_n) \circledast ((0_1, \ldots, 0_n) \circledast (b_1, \ldots, b_n)) &= (a_1, \ldots, a_n) \circledast (0_1 * b_1, \ldots, 0_n * b_n) \\ &= (a_1 * (0_1 * b_1), \ldots, a_n * (0_n * b_n)) \\ &= (b_1 * (0_1 * a_1), \ldots, b_n * (0_n * a_n)) \\ &= (b_1, \ldots, b_n) \circledast (0_1 * a_1, \ldots, 0_n * a_n) \\ &= (b_1, \ldots, b_n) \circledast ((0_1, \ldots, 0_n) \circledast (a_1, \ldots, a_n)). \end{aligned}$$

Therefore, $\prod_{i=1}^{n}\mathbf{A}_i$ is commutative.

Conversely, let $\prod_{i=1}^{n}\mathbf{A}_i$ be commutative. If $a_i, b_i \in A_i$ for all $i \in \mathrm{I}_n$, then $(a_1,\ldots,a_n), (b_1,\ldots,b_n) \in \prod_{i=1}^{n} A_i$ and $(a_1,\ldots,a_n)\circledast((0_1,\ldots,0_n)\circledast(b_1,\ldots,b_n)) = (b_1,\ldots,b_n)\circledast((0_1,\ldots,0_n)\circledast(a_1,\ldots,a_n))$.
Thus,

$$\begin{aligned}
(a_1*(0_1*b_1),\ldots,a_n*(0_n*b_n)) &= (a_1,\ldots,a_n)\circledast(0_1*b_1,\ldots,0_n*b_n)\\
&= (a_1,\ldots,a_n)\circledast((0_1,\ldots,0_n)\circledast(b_1,\ldots,b_n))\\
&= (b_1,\ldots,b_n)\circledast((0_1,\ldots,0_n)\circledast(a_1,\ldots,a_n))\\
&= (b_1,\ldots,b_n)\circledast(0_1*a_1,\ldots,0_n*a_n)\\
&= (b_1*(0_1*a_1),\ldots,b_n*(0_n*a_n)).
\end{aligned}$$

This implies that $a_i*(0_i*b_i) = b_i*(0_i*a_i)$ for all $i \in \mathrm{I}_n$. Therefore, each $\mathbf{A}_i$ is commutative. □

Theorem 2.6 *Let $\{\varphi_i\colon A_i \to B_i : i \in I_n\}$ be a family of B-homomorphisms. If φ is the map $\prod_{i=1}^{n} A_i \to \prod_{i=1}^{n} B_i$ given by $(a_1,\ldots,a_n) \mapsto (\varphi_1(a_1),\ldots,\varphi_n(a_n))$, then φ is a B-homomorphism with $ker\ \varphi = \prod_{i=1}^{n} ker\ \varphi_i$, $\varphi(\prod_{i=1}^{n} A_i) = \prod_{i=1}^{n}\varphi_i(A_i)$. Furthermore, φ is a B-monomorphism (respectively, B-epimorphism) if and only if each φ_i is a B-monomorphism (respectively, B-epimorphism).*

Proof: Let $\{\varphi_i : A_i \to B_i : i \in \mathrm{I}_n\}$ be a family of B-homomorphisms and let φ be the map $\prod_{i=1}^{n} A_i \to \prod_{i=1}^{n} B_i$ given by $(a_1,\ldots,a_n) \mapsto (\varphi_1(a_1),\ldots,\varphi_n(a_n))$. If $(a_1,\ldots,a_n), (b_1,\ldots,b_n) \in \prod_{i=1}^{n} A_i$, then

$$\begin{aligned}
\varphi((a_1,\ldots,a_n)\circledast(b_1,\ldots,b_n)) &= \varphi((a_1*b_1,\ldots,a_n*b_n))\\
&= (\varphi_1(a_1*b_1),\ldots,\varphi_n(a_n*b_n))\\
&= (\varphi_1(a_1)*\varphi_1(b_1),\ldots,\varphi_n(a_n)*\varphi_n(b_n))\\
&= (\varphi_1(a_1),\ldots,\varphi_n(a_n))\circledast(\varphi_1(b_1),\ldots,\varphi_n(b_n))\\
&= \varphi((a_1,\ldots,a_n))\circledast\varphi((b_1,\ldots,b_n)).
\end{aligned}$$

This shows that φ is a B-homomorphism. Moreover, if φ is a B-homomorphism, then each φ_i is also a B-homomorphism. Now,

$$\begin{aligned}(a_1,\dots,a_n)\in \ker\varphi &\Leftrightarrow \varphi((a_1,\dots,a_n)) = (0_1,\dots,0_n)\\ &\Leftrightarrow (\varphi_1(a_1),\dots,\varphi_n(a_n)) = (0_1,\dots,0_n)\\ &\Leftrightarrow \varphi_i(a_i) = 0_i \text{ for each } i\in \mathrm{I}_n\\ &\Leftrightarrow a_i \in \ker\varphi_i \text{ for each } i\in \mathrm{I}_n\\ &\Leftrightarrow (a_1,\dots,a_n)\in \prod_{i=1}^{n}\ker\varphi_i.\end{aligned}$$

Thus, $\ker\varphi = \prod_{i=1}^{n}\ker\varphi_i$. Let $A = \prod_{i=1}^{n} A_i$. Then

$$\begin{aligned}(b_1,\dots,b_n)\in\varphi(A) &\Leftrightarrow \exists (a_1,\dots,a_n)\in A \ni (b_1,\dots,b_n) = \varphi((a_1,\dots,a_n))\\ &\Leftrightarrow \exists (a_1,\dots,a_n)\in A \ni (b_1,\dots,b_n) = (\varphi_1(a_1),\dots,\varphi_n(a_n))\\ &\Leftrightarrow \exists a_i\in A_i \ni b_i = \varphi_i(a_i)\in\varphi(A_i) \text{ for each } i\in \mathrm{I}_n\\ &\Leftrightarrow (b_1,\dots,b_n)\in \prod_{i=1}^{n}\varphi_i(A_i).\end{aligned}$$

Thus, $\varphi(\prod_{i=1}^{n} A_i) = \prod_{i=1}^{n}\varphi_i(A_i)$.

To prove the last statement, let φ be one-to-one. If $\varphi_i(a_i) = \varphi(b_i)$ for each $i\in \mathrm{I}_n$, then

$$\begin{aligned}\varphi((a_1,\dots,a_n)) &= (\varphi_1(a_1),\dots,\varphi_n(a_n))\\ &= (\varphi_1(b_1),\dots,\varphi_n(b_n))\\ &= \varphi((b_1,\dots,b_n)).\end{aligned}$$

Since φ is one-to-one, $(a_1,\dots,a_n) = (b_1,\dots,b_n)$, that is, $a_i = b_i$ for each $i\in \mathrm{I}_n$. Therefore, φ_i is one-to-one for each $i\in \mathrm{I}_n$. Conversely, let φ_i be one-to-one for each $i\in \mathrm{I}_n$. If $\varphi((a_1,\dots,a_n)) = \varphi((b_1,\dots,b_n))$, then

$$\begin{aligned}(\varphi_1(a_1),\dots,\varphi_n(a_n)) &= \varphi((a_1,\dots,a_n))\\ &= \varphi((b_1,\dots,b_n))\\ &= (\varphi_1(b_1),\dots,\varphi_n(b_n)).\end{aligned}$$

Thus, $\varphi_i(a_i) = \varphi_i(b_i)$ for each $i\in \mathrm{I}_n$. Since each φ_i is one-to-one, $a_i = b_i$ for each $i\in \mathrm{I}_n$ and so $(a_1,\dots,a_n) = (b_1,\dots,b_n)$. Therefore, φ is one-to-one.

Finally, we show that φ is onto if and only if each φ_i is. Let φ be onto. If $b_i \in B_i$ for each $i\in \mathrm{I}_n$, then $(b_1,\dots,b_n)\in \prod_{i=1}^{n} B_i$. Since φ is onto, there exists $(a_1, \ldots, a_n) \in \prod_{i=1}^{n} A_i$ such that $(b_1, \ldots, b_n) = \varphi((a_1, \ldots, a_n)) = (\varphi_1(a_1), \ldots, \varphi_n(a_n))$, that is, $b_i = \varphi_i(a_i)$ for each $i \in \mathrm{I}_n$. Therefore, φ_i is onto for each $i \in \mathrm{I}_n$. Conversely, let φ_i be onto for each $i \in \mathrm{I}_n$. If $(b_1, \ldots, b_n) \in \prod_{i=1}^{n} B_i$, then $b_i \in B_i$ for each $i \in \mathrm{I}_n$. Since each φ_i is onto, there exists $a_i \in A_i$ such that $b_i = \varphi_i(a_i)$ for each $i \in \mathrm{I}_n$ so that $(b_1, \ldots, b_n) = (\varphi_1(a_1), \ldots, \varphi_n(a_n)) = \varphi((a_1, \ldots, a_n))$. Therefore, φ is onto and so the theorem is finally proved. □

Remark 2.7 *Let* $\{\boldsymbol{A}_i = (A_i; *, 0_i) \colon i \in I_n\}$ *and* $\{\boldsymbol{B}_i = (B_i; *, 0_i) \colon i \in I_n\}$ *be any two families of B-algebras such that* $A_i \cong B_i$ *for each* $i \in I_n$. *Then* $\prod_{i=1}^{n} A_i \cong \prod_{i=1}^{n} B_i$.

Theorem 2.8 *Let* $\{\boldsymbol{A}_i = (A_i; *, 0_i) \colon i \in I_n\}$ *be a family of B-algebras and let* J_i *be a normal subalgebra of* $\boldsymbol{A}_i$ *for each* $i \in I_n$. *Then* $\prod_{i=1}^{n} J_i$ *is a normal subalgebra of* $\prod_{i=1}^{n} \boldsymbol{A}_i$ *and* $\prod_{i=1}^{n} A_i \Big/ \prod_{i=1}^{n} J_i \cong \prod_{i=1}^{n} (A_i/J_i)$.

Proof: Let $\{\mathbf{A}_i = (A_i; *, 0_i) : i \in \mathrm{I}_n\}$ be a family of B-algebras and let J_i be a normal subalgebra of $\mathbf{A}_i$ for each $i \in \mathrm{I}_n$. Then $(0_1, \ldots, 0_n) \in \prod_{i=1}^{n} J_i$ since $0_i \in J_i$ for each $i \in \mathrm{I}_n$ and so $\prod_{i=1}^{n} J_i$ is not empty. Let $(x_1, \ldots, x_n) \circledast (y_1, \ldots, y_n), (a_1, \ldots, a_n) \circledast (b_1, \ldots, b_n) \in \prod_{i=1}^{n} J_i$. Then $(x_1 * y_1, \ldots, x_n * y_n), (a_1 * b_1, \ldots, a_n * b_n) \in \prod_{i=1}^{n} J_i$. This means that $x_i * y_i, a_i * b_i \in J_i$ for each $i \in \mathrm{I}_n$. Since each J_i is a normal subalgebra of $\mathbf{A}_i$, $(x_i * a_i) * (y_i * b_i) \in J_i$. Hence, $((x_1, \ldots, x_n) \circledast (a_1, \ldots, a_n)) \circledast ((y_1, \ldots, y_n) \circledast (b_1, \ldots, b_n)) = (x_1 * a_1, \ldots, x_n * a_n) \circledast (y_1 * b_1, \ldots, y_n * b_n) = ((x_1 * a_1) * (y_1 * b_1), \ldots, (x_n * a_n) * (y_n * b_n)) \in \prod_{i=1}^{n} J_i$. Therefore, $\prod_{i=1}^{n} J_i$ is a normal subalgebra of $\prod_{i=1}^{n} \mathbf{A}_i$.

Let $J = \prod_{i=1}^{n} J_i$ and $A = \prod_{i=1}^{n} A_i$. Define $\varphi : A/J \to \prod_{i=1}^{n} (A_i/J_i)$ given by

$\varphi((a_1,\ldots,a_n)J) = (a_1J_1,\ldots,a_nJ_n)$ for all $(a_1,\ldots,a_n)J \in A/J$. Let $(a_1,\ldots,a_n)J$, $(b_1,\ldots,b_n)J \in A/J$. If $(a_1,\ldots,a_n)J = (b_1,\ldots,b_n)J$, then $(a_1,\ldots,a_n) \sim_J (b_1,\ldots,b_n)$, that is, $(a_1*b_1,\ldots,a_n*b_n) = (a_1,\ldots,a_n) \circledast (b_1,\ldots,b_n) \in J$. Thus, $a_i * b_i \in J_i$ for all $i \in \mathrm{I}_n$, that is, $a_i \sim_{J_i} b_i$ so that $a_iJ_i = b_iJ_i$. It follows that $\varphi((a_1,\ldots,a_n)J) = (a_1J_1,\ldots,a_nJ_n) = (b_1J_1,\ldots,b_nJ_n) = \varphi((b_1,\ldots,b_n)J)$. This shows that φ is well-defined. If $(a_1,\ldots,a_n)J, (b_1,\ldots,b_n)J \in A/J$, then

$$\begin{aligned}
\varphi((a_1,\ldots,a_n)J *' (b_1,\ldots,b_n)J) &= \varphi(((a_1,\ldots,a_n) \circledast (b_1,\ldots,b_n))J)\\
&= \varphi((a_1*b_1,\ldots,a_n*b_n)J)\\
&= ((a_1*b_1)J_1,\ldots,(a_n*b_n)J_n)\\
&= (a_1J_1 *' b_1J_1,\ldots,a_nJ_n *' b_nJ_n)\\
&= (a_1J_1,\ldots,a_nJ_n) \circledast (b_1J_1,\ldots,b_nJ_n)\\
&= \varphi((a_1,\ldots,a_n)J) \circledast \varphi((b_1,\ldots,b_n)J).
\end{aligned}$$

This shows that φ is a homomorphism.
If $\varphi((a_1,\ldots,a_n)J) = \varphi((b_1,\ldots,b_n)J)$, then

$$\begin{aligned}
(a_1J_1,\ldots,a_nJ_n) &= \varphi((a_1,\ldots,a_n)J)\\
&= \varphi((b_1,\ldots,b_n)J)\\
&= (b_1J_1,\ldots,b_nJ_n).
\end{aligned}$$

Thus, $a_iJ_i = b_i/J_i$ for all $i \in \mathrm{I}_n$. Hence, $a_i \sim_{J_i} b_i$, that is, $a_i * b_i \in J_i$ for all $i \in \mathrm{I}_n$ so that $(a_1,\ldots,a_n) \circledast (b_1,\ldots,b_n) = (a_1*b_1,\ldots,a_n*b_n) \in J$. Thus, $(a_1,\ldots,a_n) \sim_J (b_1,\ldots,b_n)$ and so $(a_1,\ldots,a_n)J = (b_1,\ldots,b_n)J$. This shows that φ is one-to-one.

If $(a_1J_1,\ldots,a_nJ_n) \in \prod_{i=1}^{n}(A_i/J_i)$, then $a_i \in A_i$ for all $i \in \mathrm{I}_n$, that is, $(a_1,\ldots,a_n) \in A$. It follows that $(a_1J_1,\ldots,a_nJ_n) = \varphi((a_1,\ldots,a_n)J)$, where $(a_1,\ldots,a_n)J \in A/J$. This shows that φ is onto. Therefore, φ is a B-isomorphism, that is, $\prod_{i=1}^{n} A_i \,/\, \prod_{i=1}^{n} J_i \cong \prod_{i=1}^{n}(A_i/J_i)$. □

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Received: December 29, 2015; Published: February 5, 2016