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Ternary Semigroups in Terms of Bipolar (λ, δ)-Fuzzy Ideals

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Abstract

In this paper, we study some interesting properties of bipolar (λ, δ)-fuzzy ideals in ternary semigroups.

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1 Introduction

Lehmer [1], studied ternary algebraic structures. Sioson [2], studied the ideal theory of ternary semigroups.

Zadeh [3], introduced the concept of fuzzy sets. Fuzzy groups was first considered by Rosenfeld [4]. Lee [5] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$.

Yao introduced (λ, θ)-fuzzy normal subfields [6]. Coumaressane [7], characterized near-rings by their (λ, θ)-fuzzy quasi-ideals. Yaqoob and Ansari [8], defined the notion of bipolar (λ, δ)-fuzzy ideals in ternary semigroups.

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In this paper, we studied some new properties of bipolar (λ, δ)-fuzzy ideals in ternary semigroups and extended the notion introduced in [8].

2 Preliminaries and Basic Definitions

Throughout the paper X will be considered as a ternary semigroup unless otherwise specified.

Definition 2.1 ([1])**.** A non-empty set X whose elements are closed under the ternary operation of multiplication and satisfy the associative law defined as

$$[[abc]de] = [a[bcd]e] = [ab[cde]], \text{ for all } a, b, c, d, e \in X$$

is called a ternary semigroup.

A non-empty subset A of X is called a ternary subsemigroup of X if $AAA = A^3 \subseteq A$. By a left (right, lateral) ideal of X we mean a non-empty subset A of X such that $XXA \subseteq A$ ($AXX \subseteq A$, $XAX \subseteq A$) and an ideal is that which is a left, a right and a lateral ideal of X. A non-empty subset A of X is called a generalized bi-ideal of X if $AXAXA \subseteq A$. A ternary subsemigroup A of X is called a bi-ideal of X if $AXAXA \subseteq A$.

Definition 2.2 ([5])**.** A bipolar fuzzy set (briefly, BF-subset) $\mathcal{B}$ in X is an object having the form:

$$\mathcal{B} = \{(x, \vartheta^+_{\mathcal{B}}(x), \vartheta^-_{\mathcal{B}}(x)) \mid x \in X\}.$$

Where $\vartheta^+_{\mathcal{B}} : X \to [0, 1]$ and $\vartheta^-_{\mathcal{B}} : X \to [-1, 0]$.

The positive membership degree $\vartheta^+_{\mathcal{B}}$ denote the satisfaction degree of an element x to the property corresponding to a BF-subset $\mathcal{B}$, and the negative membership degree $\vartheta^-_{\mathcal{B}}$ denotes the satisfaction degree of x to some implicit counter property of BF-subset $\mathcal{B}$.

Let $\mathcal{X} = \{(x, \vartheta^+_{\mathcal{X}}(x), \vartheta^-_{\mathcal{X}}(x)) : \vartheta^+_{\mathcal{X}}(x) = 1 \text{ and } \vartheta^-_{\mathcal{X}}(x) = -1 \text{ for all } x \in X\}$ be a BF-subset, then $\mathcal{X} = (\vartheta^+_{\mathcal{X}}, \vartheta^-_{\mathcal{X}})$ will be carried out in operations with a BF-subset $\mathcal{B} = (\vartheta^+_{\mathcal{B}}, \vartheta^-_{\mathcal{B}})$ such that $\vartheta^+_{\mathcal{X}}$ and $\vartheta^-_{\mathcal{X}}$ will be used in collaboration with $\vartheta^+_{\mathcal{B}}$ and $\vartheta^-_{\mathcal{B}}$, respectively.

Definition 2.3. Let $\mathcal{A} = (\vartheta^+_{\mathcal{A}}, \vartheta^-_{\mathcal{A}})$, $\mathcal{B} = (\vartheta^+_{\mathcal{B}}, \vartheta^-_{\mathcal{B}})$ and $\mathcal{C} = (\vartheta^+_{\mathcal{C}}, \vartheta^-_{\mathcal{C}})$ be any three BF-subsets of a ternary semigroup $\mathcal{X}$, then the product $\mathcal{A} \circ \mathcal{B} \circ \mathcal{C} = (\vartheta^+_{\mathcal{A}\circ\mathcal{B}\circ\mathcal{C}}, \vartheta^-_{\mathcal{A}\circ\mathcal{B}\circ\mathcal{C}})$ is defined by,

$$\vartheta^+_{\mathcal{A}\circ\mathcal{B}\circ\mathcal{C}}(x) = \begin{cases} \bigvee\limits_{x=abc} \{\vartheta^+_{\mathcal{A}}(a) \wedge \vartheta^+_{\mathcal{B}}(b) \wedge \vartheta^+_{\mathcal{C}}(c)\} \\ \qquad\qquad \text{if } \exists\ a, b, c \in X \text{ such that } x = abc \\ 0 \quad \text{otherwise,} \end{cases}$$

$$\vartheta^{-}_{\mathcal{A}\circ\mathcal{B}\circ\mathcal{C}}(x) = \begin{cases} \bigwedge_{x=abc} \{\vartheta^{-}_{\mathcal{A}}(a) \vee \vartheta^{-}_{\mathcal{B}}(b) \vee \vartheta^{-}_{\mathcal{C}}(c)\} \\ \qquad\qquad \text{if } \exists\, a, b, c \in X \text{ such that } x = abc \\ 0 \quad \text{otherwise.} \end{cases}$$

3 Bipolar (λ, δ)-Fuzzy Ideals

Yaqoob and Ansari [8], introduced the notion of bipolar (λ, δ)-fuzzy ideals in ternary semigroups. In this section, we will discuss some more properties of bipolar (λ, δ)-fuzzy ideals of ternary semigroups.

Definition 3.1 ([8]). A bipolar fuzzy subset $\mathcal{B} = (\vartheta^{+}_{\mathcal{B}}, \vartheta^{-}_{\mathcal{B}})$ of a ternary semigroup X is called a bipolar (λ, δ)-fuzzy ternary subsemigroup of X if

(i) $\max\{\vartheta^{+}_{\mathcal{B}}(xyz), \lambda\} \geq \min\{\vartheta^{+}_{\mathcal{B}}(x), \vartheta^{+}_{\mathcal{B}}(y), \vartheta^{+}_{\mathcal{B}}(z), \delta\}$,

(ii) $\min\{\vartheta^{-}_{\mathcal{B}}(xyz), -\lambda\} \leq \max\{\vartheta^{-}_{\mathcal{B}}(x), \vartheta^{-}_{\mathcal{B}}(y), \vartheta^{-}_{\mathcal{B}}(z), -\delta\}$ for all $x, y, z \in X$.

Definition 3.2 ([8]). A bipolar fuzzy subset $\mathcal{B} = (\vartheta^{+}_{\mathcal{B}}, \vartheta^{-}_{\mathcal{B}})$ of a ternary semigroup X is called a bipolar (λ, δ)-fuzzy left (right, lateral) ideal of X if

(i) $\max\{\vartheta^{+}_{\mathcal{B}}(xyz), \lambda\} \geq \min\{\vartheta^{+}_{\mathcal{B}}(z), \delta\}$

$$\begin{pmatrix} \max\{\vartheta^{+}_{\mathcal{B}}(xyz), \lambda\} \geq \min\{\vartheta^{+}_{\mathcal{B}}(x), \delta\}, \\ \max\{\vartheta^{+}_{\mathcal{B}}(xyz), \lambda\} \geq \min\{\vartheta^{+}_{\mathcal{B}}(y), \delta\}, \end{pmatrix}$$

(ii) $\min\{\vartheta^{+}_{\mathcal{B}}(xyz), -\lambda\} \leq \min\{\vartheta^{+}_{\mathcal{B}}(z), -\delta\}$

$$\begin{pmatrix} \min\{\vartheta^{-}_{\mathcal{B}}(xyz), -\lambda\} \leq \max\{\vartheta^{-}_{\mathcal{B}}(x), -\delta\}, \\ \min\{\vartheta^{-}_{\mathcal{B}}(xyz), -\lambda\} \leq \max\{\vartheta^{-}_{\mathcal{B}}(y), -\delta\}, \end{pmatrix}$$

for all $x, y, z \in X$.

A bipolar fuzzy subset $\mathcal{B} = (\vartheta^{+}_{\mathcal{B}}, \vartheta^{-}_{\mathcal{B}})$ of a ternary semigroup X is called a bipolar (λ, δ)-fuzzy ideal of X if it is a bipolar (λ, δ)-fuzzy left ideal, bipolar (λ, δ)-fuzzy right ideal and bipolar (λ, δ)-fuzzy lateral ideal of X.

Definition 3.3 ([8]). A bipolar fuzzy subset $\mathcal{B} = (\vartheta^{+}_{\mathcal{B}}, \vartheta^{-}_{\mathcal{B}})$ of a ternary semigroup X is called a bipolar (λ, δ)-fuzzy generalized bi-ideal of X if

(i) $\max\{\vartheta^{+}_{\mathcal{B}}(xuyvz), \lambda\} \geq \min\{\vartheta^{+}_{\mathcal{B}}(x), \vartheta^{+}_{\mathcal{B}}(y), \vartheta^{+}_{\mathcal{B}}(z), \delta\}$,

(ii) $\min\{\vartheta^{-}_{\mathcal{B}}(xuyvz), -\lambda\} \leq \max\{\vartheta^{-}_{\mathcal{B}}(x), \vartheta^{-}_{\mathcal{B}}(y), \vartheta^{-}_{\mathcal{B}}(z), -\delta\}$, for all $x, y, z, u, v \in X$.

Definition 3.4 ([8])**.** A bipolar (λ, δ)-fuzzy ternary subsemigroup $\mathcal{B} = (\vartheta_{\mathcal{B}}^{+}, \vartheta_{\mathcal{B}}^{-})$ of a ternary semigroup X is called a bipolar (λ, δ)-fuzzy bi-ideal of X if

(i) $\max\{\vartheta_{\mathcal{B}}^{+}(xuyvz), \lambda\} \geq \min\{\vartheta_{\mathcal{B}}^{+}(x), \vartheta_{\mathcal{B}}^{+}(y), \vartheta_{\mathcal{B}}^{+}(z), \delta\}$,

(ii) $\min\{\vartheta_{\mathcal{B}}^{-}(xuyvz), -\lambda\} \leq \max\{\vartheta_{\mathcal{B}}^{-}(x), \vartheta_{\mathcal{B}}^{-}(y), \vartheta_{\mathcal{B}}^{-}(z), -\delta\}$,

for all $x, y, z, u, v \in X$.

Theorem 3.5. *Let $\mathcal{A} = (\vartheta_{\mathcal{A}}^{+}, \vartheta_{\mathcal{A}}^{-})$ be a bipolar (λ, δ)-fuzzy left ideal, $\mathcal{B} = (\vartheta_{\mathcal{B}}^{+}, \vartheta_{\mathcal{B}}^{-})$ be a bipolar (λ, δ)-fuzzy lateral ideal and $\mathcal{C} = (\vartheta_{\mathcal{C}}^{+}, \vartheta_{\mathcal{C}}^{-})$ be a bipolar (λ, δ)-fuzzy right ideal of a ternary semigroup X, then $\mathcal{A} \circ \mathcal{B} \circ \mathcal{C}$ is a bipolar (λ, δ)-fuzzy ideal of X.*

Proof. The proof is straightforward. □

Lemma 3.6. *Intersection of any family of bipolar (λ, δ)-fuzzy ternary subsemigroups (resp. bipolar (λ, δ)-fuzzy left ideals, bipolar (λ, δ)-fuzzy right ideals, bipolar (λ, δ)-fuzzy lateral ideals, bipolar (λ, δ)-fuzzy generalized bi-ideals, bipolar (λ, δ)-fuzzy bi-ideals) of a ternary semigroup X is a bipolar (λ, δ)-fuzzy ternary subsemigroup (resp. bipolar (λ, δ)-fuzzy left ideal, bipolar (λ, δ)-fuzzy right ideal, bipolar (λ, δ)-fuzzy lateral ideal, bipolar (λ, δ)-fuzzy generalized bi-ideal, bipolar (λ, δ)-fuzzy bi-ideal) of X.*

Proof. Let $\{\mathcal{A}_i\}_{i \in I}$ be a family of bipolar (λ, δ)-fuzzy left ideals of X and $x, y, z \in X$. Since each A_i is a bipolar (λ, δ)-fuzzy left ideal of X, so

$$\vartheta_{\mathcal{A}}^{+}(xyz) \vee \lambda \geq \vartheta_{\mathcal{A}}^{+}(z) \wedge \delta$$

and

$$\vartheta_{\mathcal{A}}^{-}(xyz) \wedge -\lambda \leq \vartheta_{\mathcal{A}}^{-}(z) \vee -\delta,$$

for all $i \in I$. Thus

$$\begin{aligned}
\left(\left(\bigwedge_{i \in I} \vartheta_{\mathcal{A}}^{+}\right)(xyz)\right) \vee \lambda &= \left(\bigwedge_{i \in I} (\vartheta_{\mathcal{A}}^{+}(xyz))\right) \vee \lambda \\
&= \bigwedge_{i \in I} (\vartheta_{\mathcal{A}}^{+}(xyz) \vee \lambda) \\
&\geq \bigwedge_{i \in I} (\vartheta_{\mathcal{A}}^{+}(y) \wedge \delta) \\
&= \left(\bigwedge_{i \in I} (\vartheta_{\mathcal{A}}^{+}(y))\right) \wedge \delta \\
&= \left(\bigwedge_{i \in I} \vartheta_{\mathcal{A}}^{+}\right)(y) \wedge \delta
\end{aligned}$$

and

$$\begin{aligned}\left(\left(\bigvee_{i\in I}\vartheta^-_{\mathcal{A}}\right)(xyz)\right)\wedge -\lambda &= \left(\bigvee_{i\in I}(\vartheta^-_{\mathcal{A}}(xyz))\right)\wedge -\lambda \\ &\leq \bigvee_{i\in I}(\vartheta^-_{\mathcal{A}}(xyz)\vee -\lambda) \\ &\leq \bigvee_{i\in I}(\vartheta^-_{\mathcal{A}}(y)\vee -\delta) \\ &= \left(\bigvee_{i\in I}(\vartheta^-_{\mathcal{A}}(y))\right)\vee -\delta \\ &= \left(\bigvee_{i\in I}\vartheta^-_{\mathcal{A}}\right)(y)\vee -\delta\,.\end{aligned}$$

Hence $\bigwedge_{i\in I}\mathcal{A}_i$ is a bipolar (λ, δ)-fuzzy left ideal of X. The other cases can be seen in a similar way. □

Now we prove that if $\mathcal{B} = (\vartheta^+_{\mathcal{B}}, \vartheta^-_{\mathcal{B}})$ is a bipolar (λ, δ)-fuzzy ideal of X then $\mathcal{B}^\delta_\lambda = (\vartheta^+_{\mathcal{B}^\delta_\lambda}, \vartheta^-_{\mathcal{B}^\delta_\lambda})$ is also a bipolar (λ, δ)-fuzzy ideal of X.

Definition 3.7. Let $\mathcal{B} = (\vartheta^+_{\mathcal{B}}, \vartheta^-_{\mathcal{B}})$ be a bipolar fuzzy subset of a ternary semigroup X, $\lambda, \delta \in (0, 1]$ such that $\lambda < \delta$. We define the bipolar fuzzy subset $\mathcal{B}^\delta_\lambda = (\vartheta^+_{\mathcal{B}^\delta_\lambda}, \vartheta^-_{\mathcal{B}^\delta_\lambda})$ of X as follows,

$$\vartheta^+_{\mathcal{B}^\delta_\lambda}(x) = (\vartheta^+_{\mathcal{B}}(x)\wedge\delta)\vee\lambda$$

and

$$\vartheta^-_{\mathcal{B}^\delta_\lambda}(x) = (\vartheta^-_{\mathcal{B}}(x)\vee -\delta)\wedge -\lambda$$

for all $x \in X$.

Definition 3.8. Let $\mathcal{A} = (\vartheta^+_{\mathcal{A}}, \vartheta^-_{\mathcal{A}})$ and $\mathcal{B} = (\vartheta^+_{\mathcal{B}}, \vartheta^-_{\mathcal{B}})$ be bipolar fuzzy subsets of a ternary semigroup X. Then we define,

(i) the bipolar fuzzy subset $\mathcal{A}\wedge^\delta_\lambda\mathcal{B} = (\vartheta^+_{\mathcal{A}\wedge^\delta_\lambda\mathcal{B}}, \vartheta^-_{\mathcal{A}\wedge^\delta_\lambda\mathcal{B}})$ as follows:

$$\vartheta^+_{\mathcal{A}\wedge^\delta_\lambda\mathcal{B}}(x) = \{(\vartheta^+_{\mathcal{A}}\wedge\vartheta^+_{\mathcal{B}})(x)\wedge\delta\}\vee\lambda$$

and

$$\vartheta^-_{\mathcal{A}\vee^\delta_\lambda\mathcal{B}}(x) = \{(\vartheta^-_{\mathcal{A}}\vee\vartheta^-_{\mathcal{B}})(x)\vee -\delta\}\vee -\lambda$$

(ii) the bipolar fuzzy subset $\mathcal{A} \vee_\lambda^\delta \mathcal{B} = (\vartheta^+_{\mathcal{A}\vee_\lambda^\delta\mathcal{B}}, \vartheta^-_{\mathcal{A}\wedge_\lambda^\delta\mathcal{B}})$ as follows:

$$\vartheta^+_{\mathcal{A}\vee_\lambda^\delta\mathcal{B}}(x) = \{(\vartheta^+_{\mathcal{A}} \vee \vartheta^+_{\mathcal{B}})(x) \wedge \delta\} \vee \lambda$$

and

$$\vartheta^-_{\mathcal{A}\wedge_\lambda^\delta\mathcal{B}}(x) = \{(\vartheta^-_{\mathcal{A}} \wedge \vartheta^-_{\mathcal{B}})(x) \vee -\delta\} \wedge -\lambda$$

(iii) the bipolar fuzzy subset $\mathcal{A} \circ_\lambda^\delta \mathcal{B} = (\vartheta^+_{\mathcal{A}\circ_\lambda^\delta\mathcal{B}}, \vartheta^-_{\mathcal{A}\circ_\lambda^\delta\mathcal{B}})$ as follows:

$$\vartheta^+_{\mathcal{A}\circ_\lambda^\delta\mathcal{B}}(x) = \{(\vartheta^+_{\mathcal{A}} \circ \vartheta^+_{\mathcal{B}})(x) \wedge \delta\} \vee \lambda$$

and

$$\vartheta^-_{\mathcal{A}\circ_\lambda^\delta\mathcal{B}}(x) = \{(\vartheta^-_{\mathcal{A}} \circ \vartheta^-_{\mathcal{B}})(x) \vee -\delta\} \wedge -\lambda$$

for all $x \in X$.

Lemma 3.9. *Let $\mathcal{A} = (\vartheta^+_{\mathcal{A}}, \vartheta^-_{\mathcal{A}})$, $\mathcal{B} = (\vartheta^+_{\mathcal{B}}, \vartheta^-_{\mathcal{B}})$ and $\mathcal{C} = (\vartheta^+_{\mathcal{C}}, \vartheta^-_{\mathcal{C}})$ be bipolar fuzzy subsets of a ternary semigroup X. Then the following holds.*

(i) $(\mathcal{A} \wedge_\lambda^\delta \mathcal{B} \wedge_\lambda^\delta \mathcal{C}) = (\mathcal{A}_\lambda^\delta \wedge \mathcal{B}_\lambda^\delta \wedge \mathcal{C}_\lambda^\delta)$

(ii) $(\mathcal{A} \vee_\lambda^\delta \mathcal{B} \vee_\lambda^\delta \mathcal{C}) = (\mathcal{A}_\lambda^\delta \vee \mathcal{B}_\lambda^\delta \vee \mathcal{C}_\lambda^\delta)$

(iii) $(\mathcal{A} \circ_\lambda^\delta \mathcal{B} \circ_\lambda^\delta \mathcal{C}) \geq (\mathcal{A}_\lambda^\delta \circ \mathcal{B}_\lambda^\delta \circ \mathcal{C}_\lambda^\delta)$.

If every element x of X is expressible as $x = abc$, then $(\mathcal{A} \circ_\lambda^\delta \mathcal{B} \circ_\lambda^\delta \mathcal{C}) = (\mathcal{A}_\lambda^\delta \circ \mathcal{B}_\lambda^\delta \circ \mathcal{C}_\lambda^\delta)$.

Proof. The proof is straightforward. □

XX

Theorem 3.10. *A bipolar fuzzy subset $XB = (\vartheta^+_X B, \vartheta^-_X B)$ of a ternary semigroup X is a bipolar (λ, δ)-fuzzy,*

(i) *ternary subsemigroup X if and only if $XB \circ_\lambda^\delta XB \circ_\lambda^\delta XB \subseteq XB_\lambda^\delta$,*

(ii) *left ideal of X if and only if $XX \circ_\lambda^\delta XX \circ_\lambda^\delta XB \subseteq XB_\lambda^\delta$,*

(iii) *lateral ideal of X if and only if $XX \circ_\lambda^\delta XB \circ_\lambda^\delta XX \subseteq XB_\lambda^\delta$,*

(iv) *right ideal of X if and only if $XB \circ_\lambda^\delta XX \circ_\lambda^\delta XX \subseteq XB_\lambda^\delta$,*

(v) *generalized bi-ideal of X if and only if $XB \circ_\lambda^\delta XX \circ_\lambda^\delta XB \circ_\lambda^\delta XX \circ_\lambda^\delta XB \subseteq XB_\lambda^\delta$,*

(vi) *bi-ideal of X if and only if $XB \circ_\lambda^\delta XB \circ_\lambda^\delta XB \subseteq XB$ and $XB \circ_\lambda^\delta XX \circ_\lambda^\delta XB \circ_\lambda^\delta XX \circ_\lambda^\delta XB \subseteq XB_\lambda^\delta$.*

Proof. (i) Suppose $XB = (\vartheta^+_X B, \vartheta^-_X B)$ is a bipolar (λ, δ)-fuzzy ternary subsemigroup of X and $x \in X$. Then

$$\begin{aligned}
&\vartheta^+_{XB\circ^\delta_\lambda XB\circ^\delta_\lambda XB}(x)\\
&\quad= \left\{\left(\bigvee_{x=abc}\{\vartheta^+_X B(a) \wedge \vartheta^+_X B(b) \wedge \vartheta^+_X B(c)\}\right) \wedge \delta\right\} \vee \lambda\\
&\quad= \left\{\left(\bigvee_{x=abc}\{\vartheta^+_X B(a) \wedge \vartheta^+_X B(b) \wedge \vartheta^+_X B(c) \wedge \delta\}\right) \wedge \delta\right\} \vee \lambda\\
&\quad\leq \left\{\left(\bigvee_{x=abc}\{\vartheta^+_X B(abc) \vee \lambda\}\right) \wedge \delta\right\} \vee \lambda\\
&\quad= \left\{\left(\bigvee_{x=abc}\{\vartheta^+_X B(abc) \wedge \delta\}\right) \vee \lambda\right\} \vee \lambda\\
&\quad= (\vartheta^+_X B(x) \wedge \delta) \vee \lambda\\
&\quad= (\vartheta^+_{XB^\delta_\lambda}(x)
\end{aligned}$$

and

$$\begin{aligned}
&\vartheta^-_{XB\circ^\delta_\lambda XB\circ^\delta_\lambda XB}(x)\\
&\quad= \left\{\left(\bigwedge_{x=abc}\{\vartheta^-_X B(a) \vee \vartheta^-_X B(b) \vee \vartheta^-_X B(c)\}\right) \vee -\delta\right\} \wedge -\lambda\\
&\quad= \left\{\left(\bigwedge_{x=abc}\{\vartheta^-_X B(a) \vee \vartheta^+_X B(b) \vee \vartheta^+_X B(c) \vee -\delta\}\right) \vee -\delta\right\} \wedge -\lambda\\
&\quad\geq \left\{\left(\bigwedge_{x=abc}\{\vartheta^-_X B(abc) \wedge -\lambda\}\right) \vee -\delta\right\} \wedge -\lambda
\end{aligned}$$

$$\begin{aligned}
&= \left\{\left(\bigwedge_{x=abc}\{\vartheta^-_X B(abc) \vee -\delta\}\right) \wedge -\lambda\right\} \wedge -\lambda\\
&= (\vartheta^-_X B(x) \vee -\delta) \wedge -\lambda\\
&= \vartheta^-_{XB^\delta_\lambda}(x).
\end{aligned}$$

If x is not expressible as $x = abc$ for all $a, b, c \in X$, then $\vartheta^+_{XB\circ^\delta_\lambda XB\circ^\delta_\lambda XB}(x) = \lambda \leq \vartheta^+_{XB^\delta_\lambda}(x)$ and $\vartheta^-_{XB\circ^\delta_\lambda XB\circ^\delta_\lambda XB}(x) = -\lambda \geq \vartheta^-_{XB^\delta_\lambda}(x)$. Hence $XB \circ^\delta_\lambda XB \circ^\delta_\lambda XB \subseteq XB^\delta_\delta$.

Conversely, assume that $XB \circ_\lambda^\delta XB \circ_\lambda^\delta XB \subseteq XB_\lambda^\delta$ Then for all $x, y, z \in X$, we have

$$\begin{aligned}
&\max\{\vartheta_X^+ B(xyz), \lambda\} \\
&\quad \geq (\vartheta_X^+ B(xyz) \wedge \delta) \vee \lambda \\
&\quad = \vartheta^+_{XB_\lambda^\delta}(xyz) \geq \vartheta^+_{XB\circ_\lambda^\delta XB \circ_\lambda^\delta XB}(xyz) \\
&\quad = \left\{ \left(\bigvee_{xyz=abc} \{\vartheta_X^+ B(a) \wedge \vartheta_X^+ B(b) \wedge \vartheta_X^+ B(c)\} \right) \wedge \delta \right\} \vee \lambda \\
&\quad \geq ((\vartheta_X^+ B(x) \wedge \vartheta_X^+ B(y) \wedge \vartheta_X^+ B(z)) \wedge \delta) \vee \lambda
\end{aligned}$$

and

$$\begin{aligned}
&\min\{\vartheta_X^- B(xyz), -\lambda\} \\
&\quad \leq (\vartheta_X^- B(xyz) \vee \delta) \wedge \lambda \\
&\quad = \vartheta^-_{XB_\lambda^\delta}(xyz) \leq \vartheta^-_{XB\circ_\lambda^\delta XB \circ_\lambda^\delta XB}(xyz) \\
&\quad = \left\{ \left(\bigwedge_{xyz=abc} \{\vartheta_X^- B(a) \vee \vartheta_X^- B(b) \vee \vartheta_X^- B(c)\} \right) \vee -\delta \right\} \wedge -\lambda \\
&\quad \geq ((\vartheta_X^- B(x) \vee \vartheta_X^- B(y) \vee \vartheta_X^- B(z)) \vee -\delta) \wedge -\lambda.
\end{aligned}$$

Hence $XB = (\vartheta_X^+ B, \vartheta_X^- B)$ is a bipolar (λ, δ)-fuzzy ternary subsemigroup of X.

The proofs of (ii), (iii), (iv), (v) and (vi) are similar to the proof of (i). □

Proposition 3.11. *If $XB = (\vartheta_X^+ B, \vartheta_X^- B)$ is a bipolar (λ, δ)-fuzzy ternary subsemigroup (resp. bipolar (λ, δ)-fuzzy left ideal, bipolar (λ, δ)-fuzzy right ideal, bipolar (λ, δ)-fuzzy lateral ideal, bipolar (λ, δ)-fuzzy generalized bi-ideal, bipolar (λ, δ)-fuzzy bi-ideal) of X then $XB_\lambda^\delta = (\vartheta^+_{XB_\lambda^\delta}, \vartheta^-_{XB_\lambda^\delta})$ is also a bipolar (λ, δ)-fuzzy ternary subsemi-group (resp. bipolar (λ, δ)-fuzzy left ideal, bipolar (λ, δ)-fuzzy right ideal, bipolar (λ, δ)-fuzzy lateral ideal, bipolar (λ, δ)-fuzzy generalized bi-ideal, bipolar (λ, δ)-fuzzy bi-ideal) of X.*

Proof. Suppose $XB = (\vartheta_X^+ B, \vartheta_X^- B)$ is a bipolar (λ, δ)-fuzzy ternary subsemigroup of X and $x, y, z \in X$. Then

$$\begin{aligned}
&\max\{\vartheta^{+}_{XB^{\delta}_{\lambda}}(xyz), \lambda\}\\
&\quad= ((\vartheta^{+}_{X}B(xyz) \wedge \delta) \vee \lambda) \vee \lambda\\
&\quad= (\vartheta^{+}_{X}B(xyz) \wedge \delta) \vee \lambda)\\
&\quad= (\vartheta^{+}_{X}B(xyz) \vee \lambda) \wedge (\delta \vee \lambda)\\
&\quad= (\vartheta^{+}_{X}B(xyz) \vee \lambda) \wedge \delta\\
&\quad= ((\vartheta^{+}_{X}B(xyz) \vee \lambda) \vee \lambda) \wedge \delta\\
&\quad\geq ((\vartheta^{+}_{X}B(x) \wedge \vartheta^{+}_{X}B(y) \wedge \vartheta^{+}XB(z) \wedge \delta) \vee \lambda) \wedge \delta\\
&\quad= \left(\begin{pmatrix} \vartheta^{+}_{X}B(x) \wedge \vartheta^{+}_{X}B(y) \\ \wedge\vartheta^{+}_{X}B(z) \wedge \delta \wedge \delta \wedge \delta \end{pmatrix} \vee \lambda \vee \lambda \vee \lambda\right) \wedge \delta\\
&\quad= \begin{pmatrix} ((\vartheta^{+}_{X}B(x) \wedge \delta) \vee \lambda) \wedge ((\vartheta^{+}_{X}B(y) \wedge \delta) \vee \lambda) \\ \wedge((\vartheta^{+}_{X}B(z) \wedge \delta) \vee \lambda) \end{pmatrix} \wedge \delta\\
&\quad= \vartheta^{+}_{XB^{\delta}_{\lambda}}(x) \wedge \vartheta^{+}_{XB^{\delta}_{\lambda}}(y) \wedge \vartheta^{+}_{XB^{\delta}_{\lambda}}(z) \wedge \delta\\
&\quad= \min\{\vartheta^{+}_{XB^{\delta}_{\lambda}}(x), \vartheta^{+}_{XB^{\delta}_{\lambda}}(y), \vartheta^{+}_{XB^{\delta}_{\lambda}}(z), \delta\}
\end{aligned}$$

and

$$\begin{aligned}
&\min\{\vartheta^{-}_{XB^{\delta}_{\lambda}}(xyz), -\lambda\}\\
&\quad= ((\vartheta^{-}_{X}B(xyz) \vee -\delta) \wedge -\lambda) \wedge -\lambda\\
&\quad= (\vartheta^{-}_{X}B(xyz) \vee -\delta) \wedge -\lambda)\\
&\quad= (\vartheta^{-}_{X}B(xyz) \wedge -\lambda) \vee (-\delta \wedge -\lambda)\\
&\quad= (\vartheta^{-}_{X}B(xyz) \wedge -\lambda) \vee -\delta\\
&\quad= ((\vartheta^{-}_{X}B(xyz) \wedge -\lambda) \wedge -\lambda) \vee -\delta\\
&\quad\leq ((\vartheta^{-}_{X}B(x) \vee \vartheta^{-}_{X}B(y) \vee \vartheta^{-}_{X}B(z) \vee -\delta) \wedge -\lambda) \vee -\delta\\
&\quad= \left(\begin{pmatrix} \vartheta^{-}_{X}B(x) \vee \vartheta^{-}_{X}B(y) \\ \vee\ \vartheta^{-}_{X}B(z) \vee -\delta \vee -\delta \vee -\delta \end{pmatrix} \wedge -\lambda \wedge -\lambda \wedge -\lambda\right) \vee -\delta\\
&\quad= \begin{pmatrix} ((\vartheta^{-}_{X}B(x) \vee -\delta) \wedge -\lambda) \vee ((\vartheta^{-}_{X}B(y) \vee -\delta) \wedge -\lambda) \\ \vee((\vartheta^{-}_{X}B(z) \vee -\delta) \wedge -\lambda) \end{pmatrix} \vee -\delta\\
&\quad= \vartheta^{-}_{XB^{\delta}_{\lambda}}(x) \vee \vartheta^{-}_{XB^{\delta}_{\lambda}}(y) \vee \vartheta^{-}_{XB^{\delta}_{\lambda}}(z) \vee -\delta\\
&\quad= \min\{\vartheta^{-}_{XB^{\delta}_{\lambda}}(x), \vartheta^{-}_{XB^{\delta}_{\lambda}}(y), \vartheta^{-}_{XB^{\delta}_{\lambda}}(z), -\delta\}
\end{aligned}$$

Hence $XB^{\delta}_{\lambda} = (\vartheta^{+}_{XB^{\delta}_{\lambda}}, \vartheta^{-}_{XB^{\delta}_{\lambda}})$ is a bipolar (λ, δ)-fuzzy ternary subsemigroup of X. The other cases can be seen in a similar way. □

Theorem 3.12. *Let $XA = (\vartheta^{+}_{X}A, \vartheta^{-}_{X}A)$ be a bipolar (λ, δ)-fuzzy right ideal, $XB = (\vartheta^{+}_{X}B, \vartheta^{-}_{X}B)$ be a bipolar (λ, δ)-fuzzy lateral ideal and $XC =$*

$(\vartheta_X^+ C, \vartheta_X^- C)$ be a bipolar (λ,δ)-fuzzy left ideal of X. Then $XA \circ_\lambda^\delta XB \circ_\lambda^\delta XC \subseteq XA^\delta \wedge_\lambda^\delta XB \wedge_\lambda^\delta XC$.

Proof. Let $XA = (\vartheta_X^+ A, \vartheta_X^- A)$ be a bipolar (λ,δ)-fuzzy right ideal, $XB = (\vartheta_X^+ B, \vartheta_X^- B)$ be a bipolar (λ,δ)-fuzzy lateral ideal and $XC = (\vartheta_X^+ C, \vartheta_X^- C)$ be a bipolar (λ,δ)-fuzzy left ideal of X. Then for all $x \in X$, we have

$$
\begin{aligned}
&\vartheta^+_{XA\circ_\lambda^\delta XB\circ_\lambda^\delta XC}(x)\\
&= ((\vartheta_X^+ A \circ \vartheta_X^+ B \circ \vartheta_X^+ C)(x) \wedge \delta) \vee \lambda\\
&= \left(\left(\bigvee_{x=abc} \{\vartheta_X^+ A(a) \wedge \vartheta_X^+ B(b) \wedge \vartheta_X^+ C(c)\}\right) \wedge \delta\right) \vee \lambda\\
&= \left(\bigvee_{x=abc} \{\vartheta_X^+ A(a) \wedge \vartheta_X^+ B(b) \wedge \vartheta_X^+ C(c) \wedge \delta\}\right) \wedge \lambda\\
&= \left(\bigvee_{x=abc} \left\{\begin{matrix}(\vartheta_X^+ A(a) \wedge \delta) \wedge \vartheta_X^+ B(c) \wedge \delta)\\ \wedge(\vartheta_X^+ C(c) \wedge \delta) \wedge \delta\end{matrix}\right\}\right) \vee \lambda\\
&= \left(\bigvee_{x=abc} \left\{\begin{matrix}(\vartheta_X^+ A(abc) \vee \lambda) \wedge (\vartheta_X^+ B(abc) \vee \lambda)\\ \wedge(\vartheta_X^+ C(abc) \vee \lambda) \wedge \delta\end{matrix}\right\}\right) \vee \lambda\\
&= \{(\vartheta_X^+ A(x) \vee \lambda) \wedge (\vartheta_X^+ B(x) \wedge \lambda) \wedge (\vartheta_X^+ C(x) \vee \lambda) \wedge \delta\} \vee \lambda\\
&= \{((\vartheta_X^+ A(x) \wedge \vartheta_X^+ B(x) \wedge \vartheta_X^+ C(x)) \vee \lambda) \wedge \delta\} \vee \lambda\\
&= \{(\vartheta_X^+ A \wedge \vartheta_X^+ B \wedge \vartheta_X^+ C)(x) \wedge \delta\} \vee \lambda = \vartheta^+_{XA\wedge_\lambda^\delta XB\wedge_\lambda^\delta XC}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
&\vartheta^-_{XA\circ_\lambda^\delta XB\circ_\lambda^\delta XC}(x)\\
&= ((\vartheta_X^- A \circ \vartheta_X^- B \circ \vartheta_X^- C)(x) \vee \delta) \wedge -\lambda\\
&= \left(\left(\bigwedge_{x=abc} \{\vartheta_X^- A(a) \vee \vartheta_X^- B(b) \vee \vartheta_X^- C(c)\}\right) \vee -\delta\right) \wedge -\lambda\\
&= \left(\bigwedge_{x=abc} \{\vartheta_X^- A(a) \vee \vartheta_X^- B(b) \vee \vartheta_X^- C(c) \vee -\delta\}\right) \wedge -\lambda\\
&\geq \left(\bigwedge_{x=abc} \left\{\begin{matrix}(\vartheta_X^- A(a) \vee -\delta) \vee \vartheta_X^- B(b) \vee -\delta)\\ \vee(\vartheta_X^- C(c) \vee -\delta) \vee -\lambda\end{matrix}\right\}\right) \wedge -\lambda\\
&= \left(\bigwedge_{x=abc} \left\{\begin{matrix}(\vartheta_X^- A(abc) \wedge -\lambda) \vee (\vartheta_X^- B(abc) \wedge -\lambda)\\ \vee(\vartheta_X^- C(abc) \wedge -\delta) \vee -\delta\end{matrix}\right\}\right) \wedge -\lambda\\
&= \{(\vartheta_X^- A(x) \wedge -\lambda) \vee (\vartheta_X^- B(x) \wedge -\lambda) \vee (\vartheta_X^- C(x) \wedge -\lambda) \vee -\delta\} \wedge -\lambda
\end{aligned}
$$

$$= \{((\vartheta^-_X A(x) \vee \vartheta^-_X B(x) \vee \vartheta^-_X C(x)) \vee -\lambda) \wedge -\delta\} \wedge -\lambda$$
$$= \{(\vartheta^-_X A \vee \vartheta^-_X B \vee \vartheta^-_X C)(x) \vee -\delta\} \wedge -\lambda = \vartheta^-_{XA \vee^\delta_\lambda XB \vee^\delta_\lambda XC}(x).$$

If x is not expressible as $x = abc$, then

$$\begin{aligned} \vartheta^+_{XA \circ^\delta_\lambda XB \circ^\delta_\lambda XC}(x) &= ((\vartheta^+_X A \circ \vartheta^+_X B \circ \vartheta^+_X C)(x) \wedge \delta) \vee \lambda \\ &= 0 \vee \lambda = \lambda \\ &\leq ((\vartheta^+_X A \wedge \vartheta^+_X B \wedge \vartheta^+_X C)(x) \wedge \delta) \vee \lambda \\ &= \vartheta^+_{XA \wedge^\delta_\lambda XB \wedge^\delta_\lambda XC}(x) \end{aligned}$$

and

$$\begin{aligned} \vartheta^-_{XA \circ^\delta_\lambda XB \circ^\delta_\lambda XC}(x) &= ((\vartheta^-_X A \circ \vartheta^-_X B \circ \vartheta^-_X C)(x) \vee -\delta) \wedge -\lambda \\ &= 0 \wedge -\lambda = -\lambda \\ &\geq ((\vartheta^-_X A \vee \vartheta^-_X B \vee \vartheta^-_X C)(x) \vee -\delta) \wedge -\lambda \\ &= \vartheta^-_{XA \vee^\delta_\lambda XB \vee^\delta_\lambda XC}(x). \end{aligned}$$

Hence we get $XA \circ^\delta_\lambda XB \circ^\delta_\lambda XC \subseteq XA \wedge^\delta_\lambda XB \wedge^\delta_\lambda XC$. □

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