International Journal of Algebra, Vol. 8, 2014, no. 12, 563 - 568
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ija.2014.4668

Finite Groups Having Exactly 28 Elements of Maximal Order[1]

Zhangjia Han and Ren Song

School of Applied Mathematics
Chengdu University of Information Technology
Chengdu, 610225, China

Abstract

Let G be a finite group, $M(G)$ denotes the number of elements of maximal order of G. In this note a finite group G with $M(G) = 28$ is determined.

Mathematics Subject Classification: 20D45, 20E34

Keywords: Finite groups, Classification, Number of elements of maximal order, Thompson's Conjecture

1 Introduction

All groups considered are finite. In this paper, S_p denotes Sylow p-subgroup of G, k denotes the maximal order of elements in G and $A \rtimes B$ denotes the semidirect product of A and B. For some natural number m and n, C_n^m always denotes the direct product of m cyclic groups of order n.

For convenience, in the whole paper we always set:

$$G_1 = \langle a, b, c | a^4 = b^4 = 1, a^2 = c^2, [a, c] = a^2, [b, c] = a^2, [a, b] = 1\rangle;$$

[1]This work is supported by the National Scientific Foundation of China(No:11301426) and Scientific Research Foundation of SiChuan Provincial Education Department(No:14ZA0314).

$G_2 = \langle a, b, c | a^4 = b^4 = 1, c^2 = a^2b^2, [a, c] = a^2, [b, c] = c^2, [a, b] = 1\rangle$;

$G_3 = \langle a_1, a_2, a_3, a_4, a_5, a_6 | a_1^2 = a_2^2 = a_3^2 = a_4^2 = a_5^2 = a_6^2 = 1, [a_1, a_2] = a_5, [a_1, a_3] = a_6, [a_1, a_4] = [a_1, a_5] = [a_1, a_6] = 1, [a_2, a_3] = 1, [a_2, a_4] = a_5, [a_2, a_5] = [a_2, a_6] = 1, [a_3, a_4] = [a_3, a_5] = [a_3, a_6] = 1, [a_4, a_5] = [a_4, a_6] = 1, [a_5, a_6] = 1\rangle$.

For a finite group G, we denote by $M(G)$ the number of elements of maximal order of G, and the maximal element order in G by $k = k(G)$. There is a topic related to one of Thompson's Conjectures:

Thompson's Conjecture *Let G be a finite group. For a positive integer d, define $G(d) = |\{x \in G | \text{the order of } x \text{ is } d\}|$. If S is a solvable group, $G(d) = S(d)$ for $d = 1,\ 2,\ \ldots$, then G is solvable.*

Recently, some authors have investigated this topic in several articles(see [2], [5], [6], [8]). In particular, in [1] the authors gave a complete classification of the finite group with $M(G) = 30$, and the finite group with $M(G) = 24$ are classified in [4]. In this paper, we consider a finite group G satisfying $M(G) = 28$. Our main result of this paper is:

Main Theorem *Suppose G is a finite group having exactly 28 elements of maximal order. Then G is solvable and one of the following holds:*

(1) if $k = 4$, then $G \cong Q_8 \times C_4, G_1, G_2$ or G_3;

(2) if $k = 6$, then $|G| = 2^\alpha \cdot 3^\beta$, where $2 \le \alpha \le 6$ and $1 \le \beta \le 4$;

(3) if $k = 10$, then $S_5 = C_5 \trianglelefteq G$, $|C_G(S_5)| = C_5 \times C_2^3$ and $|G/C_G(S_5)| \Big| 4$;

(4) if $k \in \{29, 58\}$, then $C_G(x) = \langle x\rangle \trianglelefteq G$. Therefore, $G/C_G(x) \lesssim Aut(C_k)$, where $o(x) = k$.

By the above theorem, we have:

Corollary *Thompson's Conjecture holds if G has exactly 28 elements of maximal order.*

2 Preliminaries

The following lemma reveals the relationship of $M(G)$ and k.

Lemma 2.1 *[8, Lemma 1] Suppose G has exactly n cyclic subgroups of order l, then the number of elements of order l (denoted by $n_l(G)$) is $n_l(G) = n\phi(l)$, where $\phi(l)$ is the Euler function of l. In particular, if n denotes the number of cyclic subgroups of G of maximal order k, then $M(G) = n\phi(k)$.*

By above lemma, we have:

Lemma 2.2 *If $M(G) = 28$ and k is maximal element order of G, then possible values of n, k and $\phi(k)$ are given in the following table:*

n	$\phi(k)$	k
28	1	2
14	2	3,4,6
7	4	5,10,12
4	7	null
2	14	null
1	28	29,58

In proving our main theorem, the following two results will be frequently used.

Lemma 2.3 *[1, Lemma 8] If the number of elements of maximal order k is m, then there exists a positive integer α such that $|G|$ divides mk^{α}.*

Lemma 2.4 *[6, Lemma 2.5] Let P be a p-group with order p^t where p is a prime, and t is a positive integer. Suppose $b \in Z(P)$, where $o(b) = p^u = k$ with u a positive integer. Then P has at least $(p-1)p^{t-1}$ elements of order k.*

Lemma 2.5 *Let G be a finite 2-group. If $exp(G) = 4$ and $M(G) = 28$. Then G is isomorphic to the following groups: $Q_8 \times C_4, G_1, G_2$ or G_3.*

proof If G is a nonabelian 2-group with $exp(G) = 4$ and every x in G of order 2 is contained in $Z(G)$. We prove that $|G| \le 64$. Suppose that $|G| > 64$. Then G has a proper subgroup $H \cong C_2 \times C_2 \times C_2 \times C_2 \times C_4$, Since every element of order 2 is contained in $Z(G)$ and $exp(G) = 4$. Obviously, $n_4(H) = 32$, a contradiction. If G is nonabelian and there exists an element of order 2 which is not contained in $Z(G)$, then $|G| \le 64$ by [4, lemma4]. If G is abelian, let $|G| = 2^t$. Then $2^{t-1} \le 28$ by Lemma 2.4. Hence $t \le 5$ and $|G| \le 32$. Therefore $|G| = 32$ or $|G| = 64$. If $|G| = 64$, then $G \cong G_3$ by [3]. If $|G| = 32$, then $G \cong Q_8 \times C_4, G_1$ or G_2 by [7, Part 3].

3 Proof of Main Theorem

By the hypothesis $M(G) = 28$, then $k \neq 2, 3$ and 5 by [1, Lemma 6]. In the following we prove our theorem case by case for the remaining possible values of k.

Case 1 $k = 4$. By Lemma 2.3, in this case G is a 2-group. By Lemma 2.5, G is isomorphic to one of the following groups: $Q_8 \times C_4, G_1, G_2$ or G_3. Thus (1) holds.

Case 2 $k = 6$. In this case $|G| = 2^{\alpha}3^{\beta}$, where $\alpha > 0$ and $\beta > 0$ by Lemma 2.3. Let x be an element of order 6. Then $|C_G(\langle x\rangle)| = 2^u \cdot 3^v$. Since there exists no element of order 9 or 4 in $C_G(x)$, we have $v \le 3$ and $u \le 3$ by $M(G) = 28$. Since G has exactly 14 cyclic subgroups of order 6, we have $|G : N_G(\langle x\rangle)| = 1, 2, 3, 4, 6, 8, 9$ or 12. If there is an element y of order 6 in G such that $|G : N_G(\langle y\rangle)| = 6, 8$ or 9, then there exists another element z of order 6 in G such that $|G : N_G(\langle z\rangle)| = 1, 2, 3, 4, 6$ or 12. That is to say, G always has an element x of order 6 such that $|G : N_G(\langle x\rangle)| = 1, 2, 3, 4, 6$ or 12. Therefore $|G|\Big|2^6 \cdot 3^4$ since $|G| = |G : N_G(\langle x\rangle)| \cdot |N_G(\langle x\rangle) : C_G(\langle x\rangle)| \cdot |C_G(\langle x\rangle)|$. Thus (2) follows.

Case 3 $k = 10$. By Lemma 2.3, we may assume that $|G| = 2^{\alpha} \cdot 5^{\beta} \cdot 7^{\gamma}$, where $\alpha, \beta > 0$ and $\gamma = 0$, or 1.

If $\gamma = 0$, then G is a $\{2, 5\}$-group and $|G| = 2^{\alpha} \cdot 5^{\beta}$. Since the number of cyclic subgroups of order 10 in G is 7, it follows that $|G : N_G(\langle x\rangle)| = 1,\ 2,\ 4$ or 5 for some element x of order 10. If $|G : N_G(\langle x\rangle)| = 4$ or 5, then there must be another element y of order 10 such that $|G : N_G(\langle y\rangle)| = 2$ or 1. Let $|C_G(\langle x\rangle)| = 2^u \cdot 5^v$. Then $u \le 3$ and $v \le 2$ since $C_G(x)$ contains at most 28 elements of order 10. And it always follows that $|N_G(\langle x\rangle)/C_G(\langle x\rangle)|\Big|4$ and $C_G(\langle x\rangle)$ is a $\{2, 5\}$-group. So we get $|G|\Big|2^6 \cdot 5^2$ since $|G| = |G : N_G(\langle x\rangle)| \cdot |N_G(\langle x\rangle) : C_G(\langle x\rangle)| \cdot |C_G(\langle x\rangle)|$. Let $S_5 \in Syl_5(G)$. Then $S_5 \le C_G(x)$. If $v = 1$ and S_5 is not normal in G, then $|G : N_G(S_5)| \ne 1$. Assume $u = 2$ or 3. Then $C_G(x)$ contains at least 8 elements of order 10. Since all elements of order 5 in G are conjugate and $|G : N_G(S_5)| \ge 6$, there are at least 48 elements of order 10 in G, a contradiction. If $u = 1$, then $C_G(\langle x\rangle) = \langle x\rangle$ contains 4 elements of order 10. Therefore $|G : N_G(S_5)| = 7$, which contradicts $\gamma = 0$. If $v = 1$ and S_5 is normal in G, then $|G/C_G(S_5)|\Big|4$ and $|C_G(S_5)| = C_5 \times C_2^2$. Thus (3) follows.

If $v = 2$, then $C_G(x)$ contains at least 24 elements of order 10 since $S_5 \le C_G(x)$. Thus $M(G) \ne 28$, a contradiction.

If $\gamma = 1$, there exists an element y of order 10 such that $|G : N_G(\langle y\rangle)| = 7$. Otherwise, as $7 \nmid |Aut(\langle y\rangle)|$, G must have an element of order 70, a contadiction. Since $|N_G(\langle y\rangle)/C_G(y)|\Big|4$, we have $S_5 \le C_G(\langle y\rangle)$, for some $S_5 \in Syl_5(G)$. By Sylow's Theorem, $|G : N_G(S_5)| = 5k' + 1 \ge 56$ since $7\ \Big||G|$, which implies that there are at least 224 elements of order 10, a contradiction.

Case 4 $k = 12$. By Lemma 2.3, we may assume that $|G| = 2^{\alpha} \cdot 3^{\beta} \cdot 7^{\gamma}$, where $\alpha, \beta > 0$ and $\gamma = 0$, or 1.

If $\gamma = 0$, then G is a $\{2, 3\}$-group and $|G| = 2^{\alpha} \cdot 3^{\beta}$. Since the number of cyclic subgroups of order 12 in G is 7, it follows that $|G : N_G(\langle x\rangle)| = 1,\ 2,\ 3,\ 4$ or 6 for some element x of order 12. If $|G : N_G(\langle x\rangle)| = 3$ or 6, then there must be another element y of order 12 such that $|G : N_G(\langle y\rangle)| = 1,\ 2$ or 4. Hence there is an element x of order 12 such that $|G : N_G(\langle x\rangle)| = 1,\ 2$ or 4. Let

$|C_G(\langle x\rangle)| = 2^u \cdot 3^v$. By Lemma 2.4, $C_G(\langle x\rangle)$ has at least 2^{u-1} elements of order 4. On the other hand, all 3-elements of $C_G(\langle x\rangle)$ is of order 3 since $C_G(\langle x\rangle)$ has no element of order 9. Hence we have $2 \cdot 2^{u-1} + 2(3^v - 1) - 4 \leq 28$ by Lemma 2.4 and our assumption. Therefore we get $1 \leq v \leq 2$ and $2 \leq u \leq 4$. So we get $|G|\Big|2^8 \cdot 3^2$ since $|G| = |G : N_G(\langle x\rangle)| \cdot |N_G(\langle x\rangle) : C_G(\langle x\rangle)| \cdot |C_G(\langle x\rangle)|$. Let $S_3 \in Syl_3(G)$. Then $S_3 \leq C_G(x)$. If $v = 1$ and S_3 is normal in G, then $n_3(G) = 2$ and hence, $28 = M(G) = n_3(G)n_2(C_G(S_3)) = 2n_2(C_G(S_3))$, so $n_2(C_G(S_3)) = 14$, which is a contradiction, because the number of the elements of order 2 is always an odd number. a contradiction. If $v = 1$ and S_3 is not normal in G, then $|G : N_G(S_3)| \neq 1$. Since $[G : N_G(S_3)] = 2s > 2, n_3(G) = 2s + 1 > 4$ and hence, $28 = M(G) = n_{12}(G) = n_3(G)n_4(C_G(S_3)) = 2^{s+1}n_4(C_G(S_3))$, which is impossible. Suppose now that $v = 2$. Then $C = C_G(\langle x\rangle) = C_4 \times C_3^2$ contains 16 elements of order 12. Choose $y \in G\backslash C$ be an element of order 12, then $C_G(y)$ also contains 16 elements of order 12. We prove that for every $t \in G\backslash C_G(x)$ with $o(t) = 12$, $C_G(t) \cap C$ contains no element of order 12. Otherwise, there is $z \in C \cap C_G(t)$ with $o(z) = 12$. Since C and $C_G(t)$ are abelian, we have $C_G(t) \leq C_G(z)$ and $C \leq C_G(z)$. Noting that all the centralizers of cyclic subgroup of order 12 are conjugate, we know that $C, C_G(t)$ and $C_G(z)$ are also conjugate. Hence $C = C_G(t) = C_G(z)$, a contradiction. Hence $C \cup C_G(t)$ contains 32 elements of order 12, a contradiction.

If $\gamma = 1$, there exists an element x of order 12 such that $|G : N_G(\langle x\rangle)| = 7$ and hence all the cyclic subgroups of order 12 are conjugate in G. Since $|N_G(\langle x\rangle)/C_G(x)|\Big|4$, we have $S_3 \leq C_G(x)$, for some $S_3 \in Syl_3(G)$. Let $C = C_G(x), S_2 \in Syl_2(G)$ and $S_3 \in Syl_3(G)$. By Lemma 2.4 and our assumption, we have $1 \leq v \leq 2$ and $2 \leq u \leq 4$. Obviously $\langle x\rangle \leq Z(C)$, the center of C. Suppose that 3^2 divides $|C|$. Then $C > \langle x\rangle$. If C is abelian, then $|C| = 2^2 \cdot 3^2$ and C contains exactly 16 elements of order 12. By the same argument as above, we can get a contradiction. If $|C| = 36$, then we can get C is abelian since $\langle x\rangle \leq Z(C)$. Therefore we may assume that $|C| > 36$ and C is not abelian. Hence $|C| \geq 72$. Obviously $C_G(S_3) = S_3 \times S_2$, where $S_2 \in Syl_2(C_G(S_3))$. Since $4 \in \pi_e(Z(C))$ and $S_3 \leq C, 4 \in \pi_e(S_2)$. So for every $y \in S_3$,

$$n_4(C_G(y)) \geq 2. \qquad (**)$$

We continue the proof in the following cases:

Subcase 1 $S_3 \trianglelefteq G$, then since $9 \notin \pi_e(G)$, S_3 is 3-elementary abelian. So $G/C_G(S_3) \lesssim GL_2(3)$. Thus $7||C_G(S_3)|$ and hence, $21 \in \pi_e(G)$, which is a contradiction.

Subcase 2 $S_3 \ntrianglelefteq G$, then then $n_3(G) \geq 24$, and $28 = n_{12}(G) \geq 24 \cdot 2$, using (**), which is impossible.

Case 5 $k \in \{29, 58\}$. Let x be an element of order k. Then $C_G(x) = \langle x\rangle \trianglelefteq G$. Therefore, $G/C_G(x) \lesssim Aut(C_k)$ and $C_G(x) \cong C_k$. Thus (5) holds.

References

[1] G. Y. Chen, W. J. Shi, Finite groups with 30 elements of maximal order, Appl Categor Struct, 16(2008), 239-247.

[2] X. L. Du, Y. Y. Jiang, On finite groups with exact 4p elements of maximal order are solvable. Chin. Ann. Math,, 5(2004), 607-612 (in Chinese).

[3] M. Hall, The Groups of Order 64, New York: Macmillan Company, 1964.

[4] Qinhui Jiang, Changguo Shao, Finite groups with 24 elements of maximal order, Front. Math. China, 4(2010), 665-678.

[5] Youyi Jiang, Finite groups with $2p^2$ elements of maximal order are solvable, Chin. Ann. Math., 1(2000), 6l-64(in Chinese).

[6] Youyi Jiang, A theorem of finite groups with 18p elements having maximal order, Alg. Coll.,2(2008), 317-329.

[7] A. D. Thomas, G. V. Wood, Group Tables, Orpington: Shiva Publishing Limited, 1980.

[8] Cheng Yang, Finite groups based on the numbers of elements of maximal order, Chin. Ann. Math., 5(1993), 561-567(in Chinese).

Received: June 5, 2014