International Journal of Algebra, Vol. 8, 2014, no. 13, 587 - 621
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ija.2014.4666

L-Functions for Burnside Rings

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Abstract

The purpose of this article is to determine the $L-$ function of the Burnside ring $B(G)$ for cyclic groups, and a functional equation for the $L-$ function of the Burnside ring for finite groups. Based on this, an explicit functional equation is given for the $L-$ function of the maximal order of the Burnside ring for cyclic groups of order p^n.

Mathematics Subject Classification: 16H15, 11S40

Keywords: Burnside rings; L-function; functional equations

1. INTRODUCTION

Throughout this paper, let G be a finite group. The Burnside ring $B\left(G\right)$ of the group G is one of the fundamental representations rings of G, namely the ring of permutation representations, and it can be studied from different points of view.

It is in many ways the universal object to consider when looking at the category of $G-$ sets. The Burnside ring is the natural framework to study the invariants attached to structured $G-$ sets, such as $G-$ posets, or more generally simplicial $G-$ sets.

$B(G)$ is a commutative ring, and one can look at is prime spectrum and primitive idempotents. This leads to various induction theorems (Artin, Conlon, Dress, Brauer).

Let X be a finite $G-$ set and let $[X]$ be its G isomorphism class.

(1.1) Definition.We define

$$B^{+}(G) := \{[X] : \quad X \quad \text{a} \quad \text{finite } G\text{−set}\},$$

which is a commutative semiring with unit, with the binary operations of disjoint union and Cartesian product, that we call $[X_1] + [X_2] := [X_1 \uplus X_2]$ and $[X_1][X_2] := [X_1 \times X_2]$. Furthermore, $0 = [\emptyset]$ and $1 = [G/G]$.

(1.2) Definition.We define Burnside ring $B(G)$ of G as the Grothendieck ring of $B^{+}(G)$.

We observe that $B(G)$ is a commutative ring, that as an abelian group, is free, generated by elements of the form $[G/H]$, where H belongs to the set of conjugacy classes of subgroups of G, which we call $\mathcal{C}(G)$. That is

$$B(G) = \bigoplus_{[H]\in\mathcal{C}(G)} \mathbb{Z}\,[G/H].$$

Let H, K be two subgroups of G. Let ${}_H\mathcal{R}_K$ be the set of representatives of induced partition in G by the double lateral classes of the form HgK, then

$$[G/H]\cdot[G/K] = \sum_{g\in {}_H\mathcal{R}_K} \left[G/(H\cap gKg^{-1})\right].$$

Let $H \leq G$ be a subgroup and X a $G-$set, we denote the set of fixed points of X under the action of H by

$$X^H = \{x \in X : \quad h\cdot x = x, \quad \forall \quad h \in H\}.$$

(1.3) Definition.We define markof H on X as the number of elements of X^H and we call it $\varphi_H(X)$.

Some of the properties that satisfy φ_H, are:

i).- $\varphi_H(X_1 \uplus X_2) = \varphi_H(X_1) + \varphi_H(X_2)$ for every X_1 and X_2 two $G-$sets.
ii).- $\varphi_H(X_1 \times X_2) = \varphi_H(X_1)\varphi_H(X_2)$ for every X_1 and X_2 two $G-$sets.
iii).- Let X_1 and X_2 be two $G-$sets. If $[X_1] = [X_2]$, then

$$\varphi_H(X_1) = \varphi_H(X_2),$$

for every $H \leq G$.
iv).- Let H and K be subgroups of G in the same conjugacy class in $\mathcal{C}(G)$, then

$$\varphi_H(X) = \varphi_K(X),$$

for every X a $G-$set.

(1.4) Definition.We define $\widetilde{B}(G) := \prod_{[H]\in\mathcal{C}(G)} \mathbb{Z}$.

Thus we have the following map:

$$\varphi : B^+(G) \to \widetilde{B}(G)$$

$$[X] \to (\varphi_H(X))_{[H]\in\mathcal{C}(G)},$$

which is a morphism of semirings that extends to a unique injective morphism of rings

$$\varphi : B(G) \to \widetilde{B}(G).$$

For further information about the Burnside Ring, see [1].

Let R be a Dedekind domain with quotient field K, and let B be a finite dimensional $K-$algebra.

For any finite dimensional $K-$space V, a full $R-$lattice in V is a finitely generated $R-$submodule Y in V such that $KY = V$, where

$$KY = \left\{ \sum \alpha_i y_i \text{ (finite sum)} : \ \alpha_i \in K, \ y_i \in Y \right\}.$$

(1.5) Definition.An $\mathbf{R}-\mathbf{order}$ in B, is a subring Λ of B, such that the center of Λ contains R and such that Λ is a full $R-$ lattice in B.

A idealof R is a full $R-$ lattice I in K. We can see that there is a non-zero element $r \in R$, such that $rI \subseteq R$.

Let $p \in \mathbb{Z}$ be a rational prime and let $\mathbb{Z}_p$ be the ring of $p-$ adic integers.

We denote the following tensor products by

$$B_p(G) = \mathbb{Z}_p \bigotimes_{\mathbb{Z}} B(G) \quad \text{and} \quad \widetilde{B}_p(G) = \mathbb{Z}_p \bigotimes_{\mathbb{Z}} \widetilde{B}(G),$$

where we have that $B_p(G)$ is a $\mathbb{Z}_p-$ order, being $\widetilde{B}_p(G)$ its maximal order.

For further information about orders, see [4, chapters 2,3].

2. L – FUNCTIONS.

Let A be a finite-dimensional semisimple algebra over a $p-$ adic rational field $\mathbb{Q}_p$, for some prime number p. Now let Λ be a $\mathbb{Z}_p-$ order in A. Let M be a left ideal of Λ, such that the index $(\Lambda : M)$ is finite. We use this index symbol in the generalised sense: if, for example, Y_1 and Y_2 are $\mathbb{Z}_p-$ lattices spanning the same $\mathbb{Q}_p-$ vector space, we put

$$(Y_1 : Y_2) = \frac{(Y_1 : Y_1 \cap Y_2)}{(Y_2 : Y_1 \cap Y_2)}.$$

The symbol $(Y_1 : Y_2)$ is therefore unambiguously defined whether or not Y_1 contains Y_2.

Let A^* be the unit group of A. Let $\psi : A^* \to S^1$ be a character (i.e., a continuous homomorphism to the unit circle in $\mathbb{C}$) of A^* which is of finite order and trivial on the subgroup $\mathrm{Aut}_\Lambda M$.

(2.1) Definition.We define local $L-$ function,as follows:

$$L_\Lambda(M, s, \psi) = \sum_N \psi(N)\,(\Lambda : N)^{-s}, \quad \mathrm{Re}\,(s) > 1,$$

where the sum extends over all left ideals N of Λ such that $N \cong M$. Here $\psi(N)$ is defined by the formula

$$\psi(N) = \psi(x),$$

for any $x \in A^*$, such that $N = Mx$. Since ψ is trivial on $\mathrm{Aut}_\Lambda M$, $\psi(N)$ is independent of the choice of x in the above formula. Then $L_\Lambda(M,s,\psi)$ is a power series in p^{-s}, with coefficients in the ring $\mathbb{Z}[\psi]$ generated by adjoining to $\mathbb{Z}$ the values of ψ, which are of course roots of unity. In cases of the commutative rings $B(G)$, $\widetilde{B}(G)$, $B_p(G)$ and $\widetilde{B}_p(G)$, the sum extends over all the ideals of finite index such that $N \cong M$, and converges uniformly on compact subsets of $\{s \in \mathbb{C} \mid \quad \mathrm{Re}(s) > 1\}$.

(2.2) Definition.We define conductorof M in Λ, as follows:

$$\{M,\Lambda\} = \{x \in A : \quad Mx \subseteq \Lambda\}.$$

Let $\Phi_{\{M,\Lambda\}}$ be the characteristic function in A of $\{M,\Lambda\}$. Now we choose a Haar measure d^*x on the unit group A^*. For measurable sets $E \subset A$, $E' \subset A^*$, it will be convenient to write:

$$\mu(E) = \int_E dx, \qquad \mu^*(E') = \int_{E'} d^*x.$$

We have that:

$$L_\Lambda(M,s,\psi) = \mu^*(\mathrm{Aut}_\Lambda M)^{-1}(\Lambda : M)^{-s} \int_{x\in A^*} \Phi_{\{M:\Lambda\}}(x)\,\psi(x)\,\|x\|_A^s\,d^*x,$$

where $\|x\|_A = (Yx : Y)$, for $x \in A^*$, which is independent of the choice of Y a full $\mathbb{Z}_p -$ lattice in A, and we observe that it is multiplicative. Furthermore, we can see that $\|x\|_A = 1$ whenever x is a unit in some $\mathbb{Z}_p -$ order in A. For further details of this result, see [2, sections 1,2].

Let W be a finite-dimensional vector space over a $p -$ adic field $\mathbb{Q}_p$.

(2.3) Definition.An application $\Phi : W \to \mathbb{C}$ is a -Bruhat function if it is locally constant with compact support, where

$$\mathrm{support}(\Phi) = \{w \in W : \quad \Phi(w) \neq 0\}.$$

(2.4) Definition.With the above notation, we define zeta-integralfor a Schwartz-Bruhat function Φ on A :

$$Z(\Phi,s,\psi) = \int_{x\in A^*} \Phi(x)\psi(x)\,\|x\|_A^s\,d^*x.$$

We observe that $Z(\Phi,s,\psi)$ admits analytic continuation to a meromorphic function on the whole complex $s -$ plane.

Let $A=\prod_{i=1}^{r}A_i$, were each A_i is a simple algebra over $\mathbb{Q}_p$, for $1\leq i\leq r$.

Let θ_i be the canonical continuous character of the additive group of A_i for $1\leq i\leq r$, defined by

$$\theta_i(x_i)=\exp\left(2\pi i\left(\mathrm{tr}_{A_i\diagup\mathbb{Q}_p}(x_i)\right)\right),\quad x_i\in A_i.$$

Here $\mathrm{tr}_{A_i\diagup\mathbb{Q}_p}$ denotes the absolute reduced trace defined by

$$\mathrm{tr}_{A_i\diagup\mathbb{Q}_p}=\mathrm{Tr}_{C_i\diagup\mathbb{Q}_p}\circ\mathrm{tr}_{A_i\diagup C_i},$$

where $\mathrm{tr}_{A_i\diagup C_i}$ is the ordinary reduced trace from A_i to its centre C_i, and $\mathrm{Tr}_{C_i\diagup\mathbb{Q}_p}$ is the field trace. Furthermore, for $c\in\mathbb{Q}_p$ we interpret $\exp(2\pi ic)$ as $\exp(2\pi ic_0)$, where $c_0\in\mathbb{Q}$ is the principal part of c.

(2.5) Definition.We define the canonical character of A by

$$\theta=\prod_{i=1}^{r}\theta_i.$$

(2.6) Definition.For a Schwartz-Bruhat function Φ on A, we define Fourier transform $\widehat{\Phi}$ of Φ by:

$$\widehat{\Phi}(y)=\int\limits_{x\in A}\Phi(x)\theta(xy)dx,\quad y\in A.$$

(2.7) Theorem (Functional equation for the zeta-integral).

Let Φ,Ψ be Schwartz-Bruhat functions on A, and let $\overline{\psi}$ be the complex conjugate of ψ. Then

$$Z(\Phi,s,\psi)Z(\widehat{\Psi},1-s,\overline{\psi})=Z(\Psi,s,\psi)Z(\widehat{\Phi},1-s,\overline{\psi}),$$

for all s. For further details of this result, see[3, section 10].

3. L – FUNCTIONS FOR $\mathbf{B_p}(\mathbf{C_{p^n}})$.

(3.1) Observation.Let $n\in\mathbb{N}$ and $\Gamma_n=B_p(C_{p^n})$ be the Burnside ring of the cyclic group of order p^n. We have that the conjugacy classes of subgroups of C_{p^n} are:

$$\mathcal{C}(C_{p^n})=\left\{C_{p^n},\ pC_{p^n},\ p^2C_{p^n},\dots,\ p^nC_{p^n}\right\},$$

whence a basis for Γ_n is:

$$1 = C_{p^n}/C_{p^n},\ a_1 = C_{p^n}/pC_{p^n}, \dots, a_n = C_{p^n}/p^n C_{p^n}.$$

Therefore, $\Gamma_n = \mathbb{Z}_p \bigoplus \mathbb{Z}_p a_1 \bigoplus \dots \bigoplus \mathbb{Z}_p a_n$. Furthermore $\widetilde{\Gamma}_n = \mathbb{Z}_p^{(n+1)}$ is its maximal order. Now, for $\mathcal{C}(C_{p^n}) = \{C_{p^n},\ pC_{p^n},\ p^2C_{p^n}, \dots,\ p^nC_{p^n}\}$, we have that φ induces the following inclusion:

$$\begin{array}{ccc} & \varphi & \\ \Gamma_n & \hookrightarrow & \widetilde{\Gamma}_n \\ X & \rightarrow & (\varphi_H(X))_{H\in\mathcal{C}(C_{p^n})} \\ & & \\ 1 & \rightarrow & (1, ..., 1) \\ a_1 & \rightarrow & (0, p, \dots, p) \\ a_2 & \rightarrow & (0, 0, p^2, \dots, p^2) \\ \vdots & \ddots & \vdots \\ a_n & \rightarrow & (0, \dots 0, p^n) \end{array}$$

Therefore:

$$\Gamma_n = \left\{(u_1, \dots, u_{n+1}) \in \mathbb{Z}_p^{(n+1)}\colon\ (u_l - u_{l-1}) \in p^{l-1}\mathbb{Z}_p \text{ for } l = 2, \dots, n+1\right\}.$$

(3.2) **L – Functions for $\mathbf{B_p\left(C_p\right)}$.**

Let $\Gamma_1 = B_p\left(C_p\right)$ be the Burnside ring of a cyclic group of order p. From (3.1) for $n = 1$, we have that:

$$\Gamma_1 = \left\{(u_1, u_2) \in \mathbb{Z}_p^2\colon\ (u_2 - u_1) \in p\mathbb{Z}_p\right\} \subseteq \widetilde{\Gamma}_1.$$

From [5, section 3], we can see that the only isomorphism classes of the fractional ideals of Γ_1 of finite index, are:

$$\Gamma_1 \text{ y } \widetilde{\Gamma}_1.$$

We choose a Haar measure d^*x on $\left(\mathbb{Q}_p^*\right)^2$ such that $d^*x = \left(d^*x_1\right)^2$, where d^*x_1 is a Haar measure on $\mathbb{Q}_p^*$, such that $\int_{\mathbb{Z}_p^*} d^*x_1 = 1$.

Therefore, we have that

$$\mu^*\left(\widetilde{\Gamma}_1^*\right)=1.$$

Let $\psi=(\psi_{t_1},\psi_{t_2}):\left(\mathbb{Q}_p^*\right)^2\to S^1$ be a continuous character of finite order, which is trivial on $\left(\mathbb{Z}_p^*\right)^2$, where

$$\psi_t:\mathbb{Q}_p^*\to S^1$$

is defined by $\psi_t(p)=\exp\left(\frac{2\pi i}{t}\right)$, for $0<t\in\mathbb{Z}$, and $\psi(b_1,b_2)=\psi_{t_1}(b_1)\,\psi_{t_2}(b_2)$, for each $(b_1,b_2)\in\left(\mathbb{Q}_p^*\right)^2$.

Remember that $\widetilde{\Gamma}_1=\mathbb{Z}_p^2$. By definition, we have:

$$\mathbf{L}_{\widetilde{\Gamma}_1}\left(\widetilde{\boldsymbol{\Gamma}}_{\mathbf{1}},\mathbf{s},\psi\right)=$$

$$\mu^*\left(\mathrm{Aut}_{\widetilde{\Gamma}_1}\widetilde{\Gamma}_1\right)^{-1}\left(\widetilde{\Gamma}_1:\widetilde{\Gamma}_1\right)^{-s}\int\limits_{x\in\left(\mathbb{Q}_p^*\right)^2}\Phi_{\left\{\widetilde{\Gamma}_1:\widetilde{\Gamma}_1\right\}}(x)\,\psi(x)\,\|x\|_{\mathbb{Q}_p^2}^s d^*x=$$

$$\mu^*\left(\widetilde{\Gamma}_1^*\right)^{-1}\int\limits_{x\in\left(\mathbb{Q}_p^*\right)^2}\Phi_{\widetilde{\Gamma}_1}(x)\,\psi(x)\,\|x\|_{\mathbb{Q}_p^2}^s d^*x=\int\limits_{x\in\left(\mathbb{Q}_p^*\right)^2\cap\mathbb{Z}_p^2}\psi(x)\,\|x\|_{\mathbb{Q}_p^2}^s d^*x=$$

$$\left[\int\limits_{x_1\in\biguplus_{n=0}^{\infty}p^n\mathbb{Z}_p^*}\psi_{t_1}(x_1)\,\|x_1\|_{\mathbb{Q}_p}^s\,d^*x_1\right]\left[\int\limits_{x_1\in\biguplus_{m=o}^{\infty}p^m\mathbb{Z}_p^*}\psi_{t_2}(x_1)\,\|x_1\|_{\mathbb{Q}_p}^s\,d^*x_1\right]=$$

$$\left(\sum_{n=0}^{\infty}\left[\psi_{t_1}(p)p^{-s}\right]^n\right)\left(\sum_{m=0}^{\infty}\left[\psi_{t_2}(p)p^{-s}\right]^m\right).$$

Therefore, we have:

$$\boldsymbol{L_{\widetilde{\Gamma}_1}\left(\widetilde{\Gamma}_1, s, \psi\right) = \frac{1}{\left(1-\psi_{t_1}(p)p^{-s}\right)\left(1-\psi_{t_2}(p)p^{-s}\right)}}.$$

1) From [5, section 3], we have that $\Gamma_1 = \Gamma_1^* \biguplus (p\mathbb{Z}_p)^2$. Furthermore, we have that:

a) $\mathrm{Aut}_{\Gamma_1}\Gamma_1 = \Gamma_1^*$,
b) $\mu^*\left(\Gamma_1^*\right)^{-1} = p-1$ and
c) $\{\Gamma_1 : \Gamma_1\} = \Gamma_1$.

Then, for Γ_1 we have that

$$\mathbf{L}_{\boldsymbol{\Gamma_1}}\left(\boldsymbol{\Gamma_1}, \mathbf{s}, \psi\right) =$$

$$\mu^*\left(\mathrm{Aut}_{\Gamma_1}\Gamma_1\right)^{-1}\left(\Gamma_1 : \Gamma_1\right)^{-s} \int\limits_{x\in\left(\mathbb{Q}_p^*\right)^2} \Phi_{\{\Gamma_1:\Gamma_1\}}(x)\,\psi(x)\,\|x\|_{\mathbb{Q}_p^2}^s d^*x =$$

$$(p-1) \int\limits_{x\in\left[\Gamma_1^*\biguplus(p\mathbb{Z}_p)^2\right]\bigcap\left(\mathbb{Q}_p^*\right)^2} \psi(x)\,\|x\|_{\mathbb{Q}_p^2}^s d^*x =$$

$$1+(p-1)\prod_{k=1}^{2}\left[\int\limits_{x_1\in\biguplus_{n_k=1}^{\infty}p^{n_k}\mathbb{Z}_p^*} \psi_{t_k}(x_1)\,\|x_1\|_{\mathbb{Q}_p}^s\, d^*x_1\right] = 1+\frac{(p-1)\,\psi_{t_1}(p)\psi_{t_2}(p)p^{-2s}}{\prod_{k=1}^{2}\left(1-\psi_{t_k}(p)p^{-s}\right)}.$$

Therefore

$$\boldsymbol{L_{\Gamma_1}\left(\Gamma_1, s, \psi\right)} =$$

$$\left[\mathbf{1}-\boldsymbol{\psi}_{t_1}(\boldsymbol{p})\boldsymbol{p}^{-s}-\boldsymbol{\psi}_{t_2}(\boldsymbol{p})\boldsymbol{p}^{-s}+\boldsymbol{\psi}_{t_1}(\boldsymbol{p})\boldsymbol{\psi}_{t_2}(\boldsymbol{p})\boldsymbol{p}^{\mathbf{1}-\mathbf{2}s}\right]\boldsymbol{L}_{\widetilde{\Gamma}_1}\left(\widetilde{\boldsymbol{\Gamma}}_{\mathbf{1}},\boldsymbol{s},\boldsymbol{\psi}\right).$$

Here are some examples:

$\boldsymbol{t_1}$	$\boldsymbol{\psi_{t_1}(p)}$	$\boldsymbol{t_2}$	$\boldsymbol{\psi_{t_2}(p)}$	$\boldsymbol{L_{\Gamma_1}(\Gamma_1, s, \psi)}$	$\boldsymbol{L_{\widetilde{\Gamma}_1}\left(\widetilde{\Gamma}_1, s, \psi\right)}$
2	-1	2	-1	$\frac{1+2p^{-s}+p^{1-2s}}{(1+p^{-s})^2}$	$\frac{1}{(1+p^{-s})^2}$
2	-1	4	i	$\frac{1+p^{-s}-ip^{-s}-ip^{1-2s}}{(1+p^{-s})(1-ip^{-s})}$	$\frac{1}{(1+p^{-s})(1-ip^{-s})}$
4	i	4	i	$\frac{1-2ip^{-s}-p^{1-2s}}{(1-ip^{-s})^2}$	$\frac{1}{(1-ip^{-s})^2}$

We can see that

$$L_{\Gamma_1}\left(\Gamma_1, 1-s, \overline{\psi}\right)=$$

$$\left[1-\psi_{t_1}^{-1}(p)p^{-1+s}-\psi_{t_2}^{-1}(p)p^{-1+s}+\psi_{t_1}^{-1}(p)\psi_{t_2}^{-1}(p)p^{-1+2s}\right]L_{\widetilde{\Gamma}_1}\left(\widetilde{\Gamma}_1, 1-s, \overline{\psi}\right).$$

Therefore, we have the following equation:

$$\frac{\boldsymbol{L}_{\Gamma_1}\left(\boldsymbol{\Gamma}_1,\boldsymbol{s},\boldsymbol{\psi}\right)}{\boldsymbol{L}_{\Gamma_1}\left(\boldsymbol{\Gamma}_1,\mathbf{1}-\boldsymbol{s},\overline{\boldsymbol{\psi}}\right)}=\left[\boldsymbol{\psi}_{t_1}(\boldsymbol{p})\boldsymbol{\psi}_{t_2}(\boldsymbol{p})\boldsymbol{p}^{\mathbf{1}-\mathbf{2}s}\right]\frac{\boldsymbol{L}_{\widetilde{\Gamma}_1}\left(\widetilde{\boldsymbol{\Gamma}}_1,\boldsymbol{s},\boldsymbol{\psi}\right)}{\boldsymbol{L}_{\widetilde{\Gamma}_1}\left(\widetilde{\boldsymbol{\Gamma}}_1,\mathbf{1}-\boldsymbol{s},\overline{\boldsymbol{\psi}}\right)}.$$

2) From [5, section 3], for $\widetilde{\Gamma}_1$, we have:

a) $\mathrm{Aut}_{\Gamma_1}\widetilde{\Gamma}_1=\widetilde{\Gamma}_1^{*}$,
b) $\left\{\widetilde{\Gamma}_1:\Gamma_1\right\}=(p\mathbb{Z}_p)^2$ and

c) $\left(\widetilde{\Gamma}_1 : \Gamma_1\right) = p.$

Thus,

$$\mathbf{L}_{\boldsymbol{\Gamma_1}}\left(\widetilde{\boldsymbol{\Gamma}}_{\mathbf{1}}, \mathbf{s}, \psi\right) =$$

$$\mu^*\left(\mathrm{Aut}_{\Gamma_1}\widetilde{\Gamma}_1\right)^{-1}\left(\Gamma_1 : \widetilde{\Gamma}_1\right)^{-s} \int\limits_{x\in\left(\mathbb{Q}_p^*\right)^2} \Phi_{\left\{\widetilde{\Gamma}_1:\Gamma_1\right\}}(x)\,\psi(x)\,\|x\|_{\mathbb{Q}_p^2}^s d^*x =$$

$$p^s \int\limits_{x\in(p\mathbb{Z}_p)^2\bigcap\left(\mathbb{Q}_p^*\right)^2} \psi(x)\,\|x\|_{\mathbb{Q}_p^2}^s d^*x =$$

$$\psi_{t_1}(p)\psi_{t_2}(p)p^{-s} \int\limits_{x\in\left(\mathbb{Q}_p^*\right)^2\cap\mathbb{Z}_p^2} \psi(x)\,\|x\|_{\mathbb{Q}_p^2}^s d^*x.$$

Therefore

$$\boldsymbol{L}_{\boldsymbol{\Gamma_1}}\left(\widetilde{\boldsymbol{\Gamma}}_{\mathbf{1}}, \boldsymbol{s}, \boldsymbol{\psi}\right) = \left[\boldsymbol{\psi_{t_1}(p)\psi_{t_2}(p)p^{-s}}\right] \boldsymbol{L}_{\widetilde{\boldsymbol{\Gamma}}_{\mathbf{1}}}\left(\widetilde{\boldsymbol{\Gamma}}_{\mathbf{1}}, \boldsymbol{s}, \boldsymbol{\psi}\right).$$

Here are some examples:

$\boldsymbol{t_1}$	$\boldsymbol{\psi_{t_1}(p)}$	$\boldsymbol{t_2}$	$\boldsymbol{\psi_{t_2}(p)}$	$\boldsymbol{L}_{\boldsymbol{\Gamma_1}}\left(\widetilde{\boldsymbol{\Gamma}}_{\mathbf{1}}, \boldsymbol{s}, \boldsymbol{\psi}\right)$	$\boldsymbol{L}_{\widetilde{\boldsymbol{\Gamma}}_{\mathbf{1}}}\left(\widetilde{\boldsymbol{\Gamma}}_{\mathbf{1}}, \boldsymbol{s}, \boldsymbol{\psi}\right)$
2	-1	2	-1	$\frac{p^{-s}}{(1+p^{-s})^2}$	$\frac{1}{(1+p^{-s})^2}$
2	-1	4	i	$\frac{-ip^{-s}}{(1+p^{-s})(1-ip^{-s})}$	$\frac{1}{(1+p^{-s})(1-ip^{-s})}$
4	i	4	i	$\frac{-p^{-s}}{(1-ip^{-s})^2}$	$\frac{1}{(1-ip^{-s})^2}$

We can see that

$$L_{\Gamma_1}\left(\widetilde{\Gamma}_1, 1-s, \overline{\psi}\right) = \left[\psi_{t_1}^{-1}(p)\psi_{t_2}^{-1}(p)p^{-1+s}\right] L_{\widetilde{\Gamma}_1}\left(\widetilde{\Gamma}_1, 1-s, \overline{\psi}\right).$$

Therefore, we have the following equation:

$$\frac{\mathbf{L}_{\mathbf{\Gamma_1}}\left(\widetilde{\mathbf{\Gamma}}_\mathbf{1}, \mathbf{s}, \psi\right)}{\mathbf{L}_{\mathbf{\Gamma_1}}\left(\widetilde{\mathbf{\Gamma}}_\mathbf{1}, \mathbf{1}-\mathbf{s}, \overline{\psi}\right)} = \left[\psi_{\mathbf{t_1}}^{\mathbf{2}}(\mathbf{p})\psi_{\mathbf{t_2}}^{\mathbf{2}}(\mathbf{p})\mathbf{p}^{\mathbf{1-2s}}\right] \frac{\mathbf{L}_{\widetilde{\mathbf{\Gamma}}_\mathbf{1}}\left(\widetilde{\mathbf{\Gamma}}_\mathbf{1}, \mathbf{s}, \psi\right)}{\mathbf{L}_{\widetilde{\mathbf{\Gamma}}_\mathbf{1}}\left(\widetilde{\mathbf{\Gamma}}_\mathbf{1}, \mathbf{1}-\mathbf{s}, \overline{\psi}\right)}.$$

(3.3) $\mathbf{L}-$ Functions for $\mathbf{B_p}\left(\mathbf{C_{p^2}}\right)$.

Let $\Gamma_2 = B_p\left(C_{p^2}\right)$ be the Burnside ring of a cyclic group of order p^2. From (3.1) for $n=2$, we have that:

$$\Gamma_2 = \left\{(u_1, u_2, u_3) \in \mathbb{Z}_p^3 \colon \ (u_2 - u_1) \in p\mathbb{Z}_p,\ (u_3 - u_2) \in p^2\mathbb{Z}_p\right\} \subseteq \widetilde{\Gamma}_2.$$

From [5, section 4], we can see that the only isomorphism classes of the fractional ideals of Γ_2 of finite index, are:

$M_1 = \Gamma_2$,
$M_2 = \widetilde{\Gamma}_2$,
$M_3 = \mathbb{Z}_p \bigoplus \Gamma_1$,
$M_4 = \Gamma_1 \bigoplus \mathbb{Z}_p$,
$M_5 = \left\{(u_1, u_2, u_3) \in \mathbb{Z}_p^3 \colon \ (u_3 - u_1) \in p\mathbb{Z}_p\right\}$,
$M_6 = \left\{(u_1, u_2, u_3) \in \mathbb{Z}_p^3 \colon \ (u_2 - u_1) \in p\mathbb{Z}_p, (u_3 - u_1) \in p\mathbb{Z}_p\right\}$,
$M_7 = \left\{(u_1, u_2, u_3) \in \mathbb{Z}_p^3 \colon \ (u_3 - u_2) \in p^2\mathbb{Z}_p\right\}$,
$M_8 = \left\{(u_1, u_2, u_3) \in \mathbb{Z}_p^3 \colon pu_1 - u_2 + u_3 \in p^2\mathbb{Z}_p\right\}$ and
$M_9 = \left\{(u_1, u_2, u_3) \in \mathbb{Z}_p^3 \colon : \ u_1 - u_2 + u_3 \in p\mathbb{Z}_p\right\}$.

We choose a Haar measure d^*x on $\left(\mathbb{Q}_p^*\right)^3$ such that $d^*x = \left(d^*x_1\right)^3$, where d^*x_1 is a Haar measure on $\mathbb{Q}_p^*$, such that $\int_{\mathbb{Z}_p^*} d^*x_1 = 1$. Therefore, we have that

$$\mu^*\left(\widetilde{\Gamma}_2^*\right) = 1.$$

Let $\psi = (\psi_{t_1}, \psi_{t_2}, \psi_{t_3}) : \left(\mathbb{Q}_p^*\right)^3 \to S^1$ be a continuous character of finite order, which is trivial on $\left(\mathbb{Z}_p^*\right)^3$, where $\psi_t : \mathbb{Q}_p^* \to S^1$ is defined by $\psi_t(p) = \exp\left(\frac{2\pi i}{t}\right)$, for $0 < t \in \mathbb{Z}$, and

$$\psi(b_1, b_2, b_3) = \psi_{t_1}(b_1)\,\psi_{t_2}(b_2)\,\psi_{t_3}(b_3),$$

for each $(b_1, b_2, b_3) \in \left(\mathbb{Q}_p^*\right)^3$.

Remember that $\widetilde{\Gamma}_2 = \mathbb{Z}_p^3$. By definition, we have that

$$\mathbf{L}_{\widetilde{\boldsymbol{\Gamma}}_2}\left(\widetilde{\boldsymbol{\Gamma}}_2, \mathbf{s}, \psi\right) =$$

$$\mu^*\left(\mathrm{Aut}_{\widetilde{\Gamma}_2}\widetilde{\Gamma}_2\right)^{-1}\left(\widetilde{\Gamma}_2 : \widetilde{\Gamma}_2\right)^{-s}\int\limits_{x\in\left(\mathbb{Q}_p^*\right)^3}\Phi_{\left\{\widetilde{\Gamma}_2:\widetilde{\Gamma}_2\right\}}(x)\,\psi(x)\,\|x\|_{\mathbb{Q}_p^3}^s d^*x =$$

$$\mu^*\left(\widetilde{\Gamma}_2^*\right)^{-1}\int\limits_{x\in\left(\mathbb{Q}_p^*\right)^3}\Phi_{\widetilde{\Gamma}_2}(x)\,\psi(x)\,\|x\|_{\mathbb{Q}_p^3}^s\,d^*x = \int\limits_{x\in\left(\mathbb{Q}_p^*\right)^3\cap\mathbb{Z}_p^3}\psi(x)\,\|x\|_{\mathbb{Q}_p^3}^s\,d^*x =$$

$$\prod_{k=1}^{3}\left[\int\limits_{x_1\in\biguplus_{n_k=0}^{\infty}p^{n_k}\mathbb{Z}_p^*}\psi_{t_k}(x_1)\,\|x_1\|_{\mathbb{Q}_p}^s\,d^*x_1\right] = \prod_{k=1}^{3}\left(\sum_{n_k=0}^{\infty}\left[\psi_{t_k}(p)p^{-s}\right]^{n_k}\right).$$

Therefore, we have:

$$\mathbf{L}_{\widetilde{\boldsymbol{\Gamma}}_2}\left(\widetilde{\boldsymbol{\Gamma}}_2, \mathbf{s}, \psi\right) = \frac{\mathbf{1}}{\left(\mathbf{1} - \psi_{\mathbf{t_1}}(\mathbf{p})\mathbf{p}^{-\mathbf{s}}\right)\left(\mathbf{1} - \psi_{\mathbf{t_2}}(\mathbf{p})\mathbf{p}^{-\mathbf{s}}\right)\left(\mathbf{1} - \psi_{\mathbf{t_3}}(\mathbf{p})\mathbf{p}^{-\mathbf{s}}\right)}.$$

1) From [5, section 4], for Γ_2, we have:

a) $\Gamma_2 = \Gamma_2^* \biguplus (p\mathbb{Z}_p \bigoplus p\Gamma_1)$
b) $\mathrm{Aut}_{\Gamma_2}\Gamma_2 = \Gamma_2^*$ and $\mu^*\left(\Gamma_2^*\right)^{-1} = p(p-1)^2$.
c) $\{\Gamma_2 : \Gamma_2\} = \Gamma_2$.

Thus

$$\mathbf{L}_{\mathbf{\Gamma_2}}\left(\mathbf{\Gamma_2}, \mathbf{s}, \psi\right) =$$

$$\mu^*\left(\mathrm{Aut}_{\Gamma_2}\Gamma_2\right)^{-1}\left(\Gamma_2 : \Gamma_2\right)^{-s} \int\limits_{x\in\left(\mathbb{Q}_p^*\right)^3} \Phi_{\{\Gamma_2:\Gamma_2\}}(x)\,\psi(x)\,\|x\|_{\mathbb{Q}_p^3}^s d^*x =$$

$$p(p-1)^2 \int\limits_{x\in\left(\mathbb{Q}_p^*\right)^3 \bigcap\left[\Gamma_2^* \biguplus (p\mathbb{Z}_p \bigoplus p\Gamma_1)\right]} \psi(x)\,\|x\|_{\mathbb{Q}_p^3}^s d^*x =$$

$$1 + p(p-1)^2\psi_{t_1}(p)\psi_{t_2}(p)\psi_{t_3}(p)p^{-3s}\left(\int\limits_{x\in\left(\mathbb{Q}_p^*\right)^3 \bigcap(\mathbb{Z}_p \bigoplus \Gamma_1)} \psi(x)\,\|x\|_{\mathbb{Q}_p^3}^s d^*x\right) =$$

$$1+\left(\frac{p(p-1)^2\psi_{t_1}(p)\psi_{t_2}(p)\psi_{t_3}(p)p^{-3s}}{\left(1-\psi_{t_1}(p)p^{-s}\right)}\right)\left(\frac{1-\left(\psi_{t_2}(p)+\psi_{t_3}(p)\right)p^{-s}+\psi_{t_2}(p)\psi_{t_3}(p)p^{1-2s}}{(p-1)\left(1-\psi_{t_2}(p)p^{-s}\right)\left(1-\psi_{t_3}(p)p^{-s}\right)}\right) =$$

$$\boldsymbol{\Bigg[1-\left(\sum_{\nu=1}^{3}\psi_{t_\nu}(p)\right)p^{-s} + \left(\psi_{t_1}(p)\psi_{t_2}(p)+\psi_{t_1}(p)\psi_{t_3}(p)+\psi_{t_2}(p)\psi_{t_3}(p)\right)p^{-2s} +}$$

$$\boldsymbol{\left(\prod_{\nu=1}^{3}\psi_{t_\nu}(p)\right)\left(p^2-p-1\right)p^{-3s} + \left(\prod_{\nu=1}^{3}\psi_{t_\nu}(p)\right)\left(\psi_{t_2}(p)+\psi_{t_3}(p)\right)\left(p-p^2\right)p^{-4s} +}$$

$$\boldsymbol{\psi_{t_1}(p)\psi_{t_2}^2(p)\psi_{t_3}^2(p)\left(p^3-p^2\right)p^{-5s}\Big]\,L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, s, \psi\right)}.$$

Here are some examples:

t_k	$\psi_{t_k}(p)$	$L_{\Gamma_2}(\Gamma_2, s, \psi)$
2	-1	$\frac{1+3p^{-s}+3p^{-2s}-\left(p^2-p-1\right)p^{-3s}+2\left(p-p^2\right)p^{-4s}-\left(p^3-p^2\right)p^{-5s}}{\left(1+p^{-s}\right)^3}$
4	i	$\frac{1-3ip^{-s}-3p^{-2s}-i\left(p^2-p-1\right)p^{-3s}+2\left(p-p^2\right)p^{-4s}+i\left(p^3-p^2\right)p^{-5s}}{\left(1-ip^{-s}\right)^3}.$

2) From [5, section 4], for $\widetilde{\Gamma}_2$, we have:

a) $\mathrm{Aut}_{\Gamma_2}\widetilde{\Gamma}_2 = \widetilde{\Gamma}_2^*$ and $\mu^*\left(\widetilde{\Gamma}_2^*\right)^{-1} = 1$.
b) $\left\{\widetilde{\Gamma}_2 : \Gamma_2\right\} = (p, p^2, p^2)\,\widetilde{\Gamma}_2$.
c) $\left(\widetilde{\Gamma}_2 : \Gamma_2\right) = p^3$.

Thus

$$\mathbf{L}_{\Gamma_2}\left(\widetilde{\Gamma}_2, s, \psi\right) =$$

$$\mu^*\left(\mathrm{Aut}_{\Gamma_2}\widetilde{\Gamma}_2\right)^{-1}\left(\Gamma_2 : \widetilde{\Gamma}_2\right)^{-s} \int\limits_{x\in\left(\mathbb{Q}_p^*\right)^3} \Phi_{\left\{\widetilde{\Gamma}_2:\Gamma_2\right\}}(x)\,\psi(x)\,\|x\|_{\mathbb{Q}_p^3}^s d^*x =$$

$$p^{3s} \int\limits_{x\in\left(\mathbb{Q}_p^*\right)^3 \bigcap (p,p^2,p^2)\widetilde{\Gamma}_2} \psi(x)\,\|x\|_{\mathbb{Q}_p^3}^s d^*x =$$

$$\psi_{t_1}(p)\psi_{t_2}^2(p)\psi_{t_3}^2(p)p^{-2s} \int\limits_{x\in\left(\mathbb{Q}_p^*\right)^3 \bigcap \widetilde{\Gamma}_2} \psi(x)\,\|x\|_{\mathbb{Q}_p^3}^s d^*x.$$

Therefore

$$\boldsymbol{L_{\Gamma_2}\left(\widetilde{\Gamma}_2, s, \psi\right) = \left[\psi_{t_1}(p)\psi_{t_2}^2(p)\psi_{t_3}^2(p)p^{-2s}\right] L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, s, \psi\right)}.$$

Here are some examples:

$\boldsymbol{t_k}$	$\psi_{\mathbf{t_k}}(\mathbf{p})$	$\mathbf{L}_{\mathbf{\Gamma_2}}\left(\widetilde{\mathbf{\Gamma}}_\mathbf{2}, \mathbf{s}, \psi\right)$
2	-1	$\frac{-p^{-2s}}{(1+p^{-s})^3}$
4	i	$\frac{ip^{-2s}}{(1-ip^{-s})^3}$.

We can see that

$$L_{\Gamma_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right) = \left[\psi_{t_1}^{-1}(p)\psi_{t_2}^{-2}(p)\psi_{t_3}^{-2}(p)p^{-2+2s}\right] L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right).$$

Therefore, we have the following equation:

$$\boldsymbol{\frac{L_{\Gamma_2}\left(\widetilde{\Gamma}_2, s, \psi\right)}{L_{\Gamma_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right)} = \left[\psi_{t_1}^2(p)\psi_{t_2}^4(p)\psi_{t_3}^4(p)p^{2-4s}\right] \frac{L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, s, \psi\right)}{L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right)}}.$$

3) From [5, section 4], for $M_3 = \mathbb{Z}_p \bigoplus \Gamma_1$, we have that:

a) $\mathrm{Aut}_{\Gamma_2} M_3 = M_3^*$ and $\mu^*\left(M_3^*\right)^{-1} = p-1$.
b) $\{M_3 : \Gamma_2\} = (p,p,p)\, M_3$.
c) $(M_3 : \Gamma_2) = p^2$.

Thus,

$$\boldsymbol{L_{\Gamma_2}\left(M_3, s, \psi\right) =}$$

$$\mu^* \left(\mathrm{Aut}_{\Gamma_2} M_3\right)^{-1} \left(\Gamma_2 : M_3\right)^{-s} \int\limits_{x \in \left(\mathbb{Q}_p^*\right)^3} \Phi_{\{M_3 : \Gamma_2\}}\left(x\right) \psi(x) \left\|x\right\|_{\mathbb{Q}_p^3}^s d^*x =$$

$$(p-1)p^{2s} \int\limits_{x \in \left(\mathbb{Q}_p^*\right)^3 \bigcap (p,p,p) M_3} \psi(x) \left\|x\right\|_{\mathbb{Q}_p^3}^s d^*x =$$

$$(p-1)\psi_{t_1}(p)\psi_{t_2}(p)\psi_{t_3}(p)p^{-s} \left(\frac{1-\left(\psi_{t_2}(p)+\psi_{t_3}(p)\right)p^{-s}+\psi_{t_2}(p)\psi_{t_3}(p)p^{1-2s}}{(p-1)\left(1-\psi_{t_1}(p)p^{-s}\right)\left(1-\psi_{t_2}(p)p^{-s}\right)\left(1-\psi_{t_3}(p)p^{-s}\right)} \right).$$

Therefore

$$\boldsymbol{L_{\Gamma_2}\left(M_3, s, \psi\right) =}$$

$$\boldsymbol{\left[\prod_{k=1}^{3} \psi_{t_k}(p)\right] p^{-s} \left[1-\left(\psi_{t_2}(p)+\psi_{t_3}(p)\right)p^{-s}+\psi_{t_2}(p)\psi_{t_3}(p)p^{1-2s}\right] L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, s, \psi\right).}$$

Here are some examples:

$\boldsymbol{t_k}$	$\psi_{\mathbf{t_k}}(\mathbf{p})$	$\mathbf{L_{\Gamma_2}\left(M_3, s, \psi\right)}$
2	-1	$\frac{\left[-p^{-s}\left(1+2p^{-s}+p^{1-2s}\right)\right]}{\left(1+p^{-s}\right)^3}$
4	i	$\frac{\left[-ip^{-s}\left(1-2ip^{-s}-p^{1-2s}\right)\right]}{\left(1-ip^{-s}\right)^3}.$

We can see that

$$L_{\Gamma_2}\left(M_3, 1-s, \overline{\psi}\right) =$$

$$\left[\frac{p^{-1+s}}{\prod_{k=1}^{3} \psi_{t_k}(p)}\right] \left[1-\left(\psi_{t_2}^{-1}(p)+\psi_{t_3}^{-1}(p)\right)p^{-1+s}+\psi_{t_2}^{-1}(p)\psi_{t_3}^{-1}(p)p^{-1+2s}\right] L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right).$$

Therefore, we have the following equation:

$$\frac{\boldsymbol{L_{\Gamma_2}\left(M_3, s, \psi\right)}}{\boldsymbol{L_{\Gamma_2}\left(M_3, 1-s, \overline{\psi}\right)}} = \boldsymbol{\left[\psi_{t_1}^2(p)\psi_{t_2}^3(p)\psi_{t_3}^3(p)p^{2-4s}\right] \frac{L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, s, \psi\right)}{L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right)}}.$$

4) From [5, section 4], for $M_4 = \Gamma_1 \bigoplus \mathbb{Z}_p$ we have that:

a) $\mathrm{Aut}_{\Gamma_2} M_4 = M_4^*$ and $\mu^*\left(M_4^*\right)^{-1} = p-1$.
b) $\{M_4 : \Gamma_2\} = (p, p^2, p^2)\,\widetilde{\Gamma}_2$
c) $(M_4 : \Gamma_2) = p^2$.

Thus:

$$\boldsymbol{L_{\Gamma_2}\left(M_4, s, \psi\right) =}$$

$$\mu^*\left(\mathrm{Aut}_{\Gamma_2} M_4\right)^{-1}\left(\Gamma_2 : M_4\right)^{-s} \int\limits_{x \in \left(\mathbb{Q}_p^*\right)^3} \Phi_{\{M_4:\Gamma_2\}}(x)\,\psi(x)\,\|x\|_{\mathbb{Q}_p^3}^s d^*x =$$

$$(p-1)p^{2s} \int\limits_{x \in \left(\mathbb{Q}_p^*\right)^3 \bigcap (p,p^2,p^2)\widetilde{\Gamma}_2} \psi(x)\,\|x\|_{\mathbb{Q}_p^3}^s d^*x =$$

$$(p-1)\psi_{t_1}(p)\psi_{t_2}^2(p)\psi_{t_3}^2(p)p^{-3s}\left(\int\limits_{x \in \left(\mathbb{Q}_p^*\right)^3 \bigcap \widetilde{\Gamma}_2} \psi(x)\,\|x\|_{\mathbb{Q}_p^3}^s d^*x\right).$$

Therefore

$$\boldsymbol{L_{\Gamma_2}\left(M_4, s, \psi\right) = \left[(p-1)\psi_{t_1}(p)\psi_{t_2}^2(p)\psi_{t_3}^2(p)p^{-3s}\right] L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, s, \psi\right)}.$$

Here are some examples:

$\boldsymbol{t_k}$	$\psi_{\mathbf{t_k}}(\mathbf{p})$	$\mathbf{L_{\Gamma_2}\left(M_4, s, \psi\right)}$
2	-1	$\frac{\left[-(p-1)p^{-3s}\right]}{(1+p^{-s})^3}$
4	i	$\frac{\left[(p-1)ip^{-3s}\right]}{(1-ip^{-s})^3}$.

We can see that

$$L_{\Gamma_2}\left(M_4, 1-s, \overline{\psi}\right) = \left[(p-1)\psi_{t_1}^{-1}(p)\psi_{t_2}^{-2}(p)\psi_{t_3}^{-2}(p)p^{-3+3s}\right] L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right).$$

Therefore, we have the following equation:

$$\frac{L_{\Gamma_2}\left(M_4, s, \psi\right)}{L_{\Gamma_2}\left(M_4, 1-s, \overline{\psi}\right)} = \left[\psi_{t_1}^{2}(p)\psi_{t_2}^{4}(p)\psi_{t_3}^{4}(p)p^{3-6s}\right] \frac{L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, s, \psi\right)}{L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right)}.$$

5) From [5, section 4], for $M_5 = \left\{(u_1, u_2, u_3) \in \mathbb{Z}_p^3\colon\ (u_3 - u_1) \in p\mathbb{Z}_p\right\}$, we have:

a) $\mathrm{Aut}_{\Gamma_2} M_5 = M_5^*$ and $\mu^*\left(M_5^*\right)^{-1} = p-1$.
b) $\{M_5 : \Gamma_2\} = (p, p^2, p^2)\,\widetilde{\Gamma}_2$.
c) $(M_5 : \Gamma_2) = p^2$.

Thus:

$$\boldsymbol{L_{\Gamma_2}\left(M_5, s, \psi\right) =}$$

$$\mu^*\left(\mathrm{Aut}_{\Gamma_2} M_5\right)^{-1}\left(\Gamma_2 : M_5\right)^{-s} \int\limits_{x\in\left(\mathbb{Q}_p^*\right)^3} \Phi_{\{M_5:\Gamma_2\}}\left(x\right)\psi(x)\left\|x\right\|_{\mathbb{Q}_p^3}^{s} d^*x =$$

$$(p-1)p^{2s} \int_{x\in\left(\mathbb{Q}_p^*\right)^3 \bigcap (p,p^2,p^2)\widetilde{\Gamma}_2} \psi(x) \left\|x\right\|_{\mathbb{Q}_p^3}^{s} d^*x =$$

$$(p-1)\psi_{t_1}(p)\psi_{t_2}^2(p)\psi_{t_3}^2(p)p^{-3s}\left(\int_{x\in\left(\mathbb{Q}_p^*\right)^3 \bigcap \widetilde{\Gamma}_2} \psi(x) \left\|x\right\|_{\mathbb{Q}_p^3}^{s} d^*x\right).$$

Therefore

$$\boldsymbol{L_{\Gamma_2}\left(M_5, s, \psi\right) = \left[(p-1)\psi_{t_1}(p)\psi_{t_2}^2(p)\psi_{t_3}^2(p)p^{-3s}\right] L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, s, \psi\right).}$$

Here are some examples:

t_k	$\psi_{\mathbf{t_k}}(\mathbf{p})$	$\mathbf{L_{\Gamma_2}\left(M_5, s, \psi\right)}$
2	-1	$\frac{\left[-(p-1)p^{-3s}\right]}{\left(1+p^{-s}\right)^3}$
4	i	$\frac{\left[(p-1)ip^{-3s}\right]}{\left(1-ip^{-s}\right)^3}$.

We can see that

$$L_{\Gamma_2}\left(M_5, 1-s, \overline{\psi}\right) = \left[(p-1)\psi_{t_1}^{-1}(p)\psi_{t_2}^{-2}(p)\psi_{t_3}^{-2}(p)p^{-3+3s}\right] L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right),$$

Therefore, we have the following equation:

$$\boldsymbol{\frac{L_{\Gamma_2}\left(M_5, s, \psi\right)}{L_{\Gamma_2}\left(M_5, 1-s, \overline{\psi}\right)} = \left[\psi_{t_1}^2(p)\psi_{t_2}^4(p)\psi_{t_3}^4(p)p^{3-6s}\right] \frac{L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, s, \psi\right)}{L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right)}.}$$

6) From [5, section 4], for $M_6 = \left\{(u_1, u_2, u_3) \in \mathbb{Z}_p^3\colon\ (u_2-u_1), (u_3-u_1) \in p\mathbb{Z}_p\right\}$, we have that:

a) $\mathrm{Aut}_{\Gamma_2}(M_6) = M_6^*$ and $\mu^*(M_6^*)^{-1} = (p-1)^2$.
b) $\{M_6 : \Gamma_2\} = (p,p,p)\, M_3$.
c) $(M_6 : \Gamma_2) = p$.

Thus,

$$\boldsymbol{L_{\Gamma_2}(M_6, s, \psi) =}$$

$$\mu^*(\mathrm{Aut}_{\Gamma_2} M_6)^{-1} (\Gamma_2 : M_6)^{-s} \int\limits_{x \in \left(\mathbb{Q}_p^*\right)^3} \Phi_{\{M_6:\Gamma_2\}}(x)\, \psi(x)\, \|x\|_{\mathbb{Q}_p^3}^s d^*x =$$

$$(p-1)^2 p^s \int\limits_{x \in \left(\mathbb{Q}_p^*\right)^3 \bigcap (p,p,p) M_3} \psi(x)\, \|x\|_{\mathbb{Q}_p^3}^s d^*x =$$

$$(p-1)^2 \psi_{t_1}(p)\psi_{t_2}(p)\psi_{t_3}(p) p^{-2s} \left(\frac{1 - (\psi_{t_2}(p) + \psi_{t_3}(p))\, p^{-s} + \psi_{t_2}(p)\psi_{t_3}(p) p^{1-2s}}{(p-1)\left(1-\psi_{t_1}(p)p^{-s}\right)\left(1-\psi_{t_2}(p)p^{-s}\right)\left(1-\psi_{t_3}(p)p^{-s}\right)} \right).$$

Therefore

$$\boldsymbol{L_{\Gamma_2}(M_6, s, \psi) =}$$

$$\boldsymbol{\left[\frac{(p-1) \prod_{k=1}^3 \psi_{t_k}(p)}{p^{2s}} \right] \left[1 - (\psi_{t_2}(p) + \psi_{t_3}(p))\, p^{-s} + \psi_{t_2}(p)\psi_{t_3}(p) p^{1-2s} \right] L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, s, \psi\right).}$$

Here are some examples:

$\boldsymbol{t_k}$	$\psi_{\mathbf{t_k}}(\mathbf{p})$	$\mathbf{L}_{\boldsymbol{\Gamma_2}}(\mathbf{M_6}, \mathbf{s}, \psi)$
2	-1	$\frac{\left[-(p-1)p^{-2s}\left(1+2p^{-s}+p^{1-2s}\right)\right]}{(1+p^{-s})^3}$
4	i	$\frac{\left[-i(p-1)p^{-2s}\left(1-2ip^{-s}-p^{1-2s}\right)\right]}{(1-ip^{-s})^3}$.

We can see that

$$L_{\Gamma_2}\left(M_6, 1-s, \overline{\psi}\right) =$$

$$\left[\frac{(p-1)\,p^{-2+2s}}{\prod_{k=1}^{3}\psi_{t_k}(p)}\right]\left[1-\left(\psi_{t_2}^{-1}(p)+\psi_{t_3}^{-1}(p)\right)p^{-1+s}+\psi_{t_2}^{-1}(p)\psi_{t_3}^{-1}(p)p^{-1+2s}\right]L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right),$$

Therefore, we have the following equation:

$$\boldsymbol{\frac{L_{\Gamma_2}\left(M_6, s, \psi\right)}{L_{\Gamma_2}\left(M_6, 1-s, \overline{\psi}\right)} = \left[\psi_{t_1}^2(p)\psi_{t_2}^3(p)\psi_{t_3}^3(p)p^{3-6s}\right]\frac{L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, s, \psi\right)}{L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right)}.}$$

7) From [5, section 4], for $M_7 = \left\{(u_1,u_2,u_3)\in\mathbb{Z}_p^3\colon\ (u_3-u_2)\in p^2\mathbb{Z}_p\right\}$, we have that:

a) $\mathrm{Aut}_{\Gamma_2}(M_7) = M_7^*$ and $\mu^*\left(M_7^*\right)^{-1} = p\,(p-1)$.
b) $\{M_7 : \Gamma_2\} = (p,p,p)\,M_3$.
c) $(M_7 : \Gamma_2) = p$.

Thus,

$$\boldsymbol{L_{\Gamma_2}\left(M_7, s, \psi\right) =}$$

$$\mu^*\left(\mathrm{Aut}_{\Gamma_2}M_7\right)^{-1}\left(\Gamma_2 : M_7\right)^{-s}\int\limits_{x\in\left(\mathbb{Q}_p^*\right)^3}\Phi_{\{M_7:\Gamma_2\}}(x)\,\psi(x)\,\|x\|_{\mathbb{Q}_p^3}^s d^*x =$$

$$(p-1)pp^s\int\limits_{x\in\left(\mathbb{Q}_p^*\right)^3\bigcap(p,p,p)M_3}\psi(x)\,\|x\|_{\mathbb{Q}_p^3}^s d^*x =$$

$$(p-1)\psi_{t_1}(p)\psi_{t_2}(p)\psi_{t_3}(p)p^{1-2s}\left(\frac{1-\left(\psi_{t_2}(p)+\psi_{t_3}(p)\right)p^{-s}+\psi_{t_2}(p)\psi_{t_3}(p)p^{1-2s}}{(p-1)\left(1-\psi_{t_1}(p)p^{-s}\right)\left(1-\psi_{t_2}(p)p^{-s}\right)\left(1-\psi_{t_3}(p)p^{-s}\right)}\right).$$

Therefore

$$\boldsymbol{L_{\Gamma_2}\left(M_7, s, \psi\right) =}$$

$$\boldsymbol{\left[\prod_{k=1}^{3} \psi_{t_k}(p)\right] p^{1-2s}\left[1-\left(\psi_{t_2}(p)+\psi_{t_3}(p)\right) p^{-s}+\psi_{t_2}(p)\psi_{t_3}(p)p^{1-2s}\right] L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, s, \psi\right).}$$

Here are some examples:

t_k	$\psi_{\mathbf{t_k}}(\mathbf{p})$	$\mathbf{L_{\Gamma_2}}\left(\mathbf{M_7}, \mathbf{s}, \psi\right)$
2	-1	$\frac{\left[-p^{1-2s}\left(1+2p^{-s}+p^{1-2s}\right)\right]}{\left(1+p^{-s}\right)^3}$
4	i	$\frac{\left[-ip^{1-2s}\left(1-2ip^{-s}-p^{1-2s}\right)\right]}{\left(1-ip^{-s}\right)^3}.$

We can see that

$$L_{\Gamma_2}\left(M_7, 1-s, \overline{\psi}\right) =$$

$$\left[\frac{p^{-1+2s}}{\prod_{k=1}^{3} \psi_{t_k}(p)}\right]\left[1-\left(\psi_{t_2}^{-1}(p)+\psi_{t_3}^{-1}(p)\right) p^{-1+s}+\psi_{t_2}^{-1}(p)\psi_{t_3}^{-1}(p)p^{-1+2s}\right] L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right).$$

Therefore, we have the following equation:

$$\boldsymbol{\frac{L_{\Gamma_2}\left(M_7, s, \psi\right)}{L_{\Gamma_2}\left(M_7, 1-s, \overline{\psi}\right)} = \left[\psi_{t_1}^2(p)\psi_{t_2}^3(p)\psi_{t_3}^3(p)p^{3-6s}\right] \frac{L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, s, \psi\right)}{L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right)}.}$$

8) From [5, section 4], for $M_8 = \left\{(u_1, u_2, u_3) \in \mathbb{Z}_p^3\colon\ pu_1 - u_2 + u_3 \in p^2\mathbb{Z}_p\right\}$, we have that:

a) $\mathrm{Aut}_{\Gamma_2}\left(M_8\right) = \Gamma_2^*$ and $\mu^*\left(\Gamma_2^*\right)^{-1} = p\left(p-1\right)^2.$
b) $\{M_8 : \Gamma_2\} = (p, p, p)\, M_3.$
c) $(M_8 : \Gamma_2) = p.$

Thus,

$$\boldsymbol{L_{\Gamma_2}\left(M_8, s, \psi\right) =}$$

$$\mu^*\left(\mathrm{Aut}_{\Gamma_2}M_8\right)^{-1}\left(\Gamma_2:M_8\right)^{-s}\int\limits_{x\in\left(\mathbb{Q}_p^*\right)^3}\Phi_{\{M_8:\Gamma_2\}}(x)\,\psi(x)\,\|x\|_{\mathbb{Q}_p^3}^s d^*x=$$

$$(p-1)^2p^{1+s}\int\limits_{x\in\left(\mathbb{Q}_p^*\right)^3\bigcap(p,p,p)M_3}\psi(x)\,\|x\|_{\mathbb{Q}_p^3}^s d^*x=$$

$$(p-1)^2\left[\prod_{k=1}^{3}\psi_{t_k}(p)\right]p^{1-2s}\left(\frac{1-\left(\psi_{t_2}(p)+\psi_{t_3}(p)\right)p^{-s}+\psi_{t_2}(p)\psi_{t_3}(p)p^{1-2s}}{(p-1)\left(1-\psi_{t_1}(p)p^{-s}\right)\left(1-\psi_{t_2}(p)p^{-s}\right)\left(1-\psi_{t_3}(p)p^{-s}\right)}\right).$$

Therefore

$$\boldsymbol{L_{\Gamma_2}\left(M_8,s,\psi\right)=}$$

$$\boldsymbol{\left[\frac{(p-1)\prod_{k=1}^{3}\psi_{t_k}(p)}{p^{-1+2s}}\right]\left[1-\left(\psi_{t_2}(p)+\psi_{t_3}(p)\right)p^{-s}+\psi_{t_2}(p)\psi_{t_3}(p)p^{1-2s}\right]L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2,s,\psi\right).}$$

Here are some examples:

$\boldsymbol{t_k}$	$\psi_{\mathbf{t_k}}(\mathbf{p})$	$\mathbf{L_{\Gamma_2}}\left(\mathbf{M_8},\mathbf{s},\psi\right)$
2	-1	$\frac{\left[-(p-1)p^{1-2s}\left(1+2p^{-s}+p^{1-2s}\right)\right]}{\left(1+p^{-s}\right)^3}$
4	i	$\frac{\left[-i(p-1)p^{1-2s}\left(1-2ip^{-s}-p^{1-2s}\right)\right]}{\left(1-ip^{-s}\right)^3}$.

We can see that

$$L_{\Gamma_2}\left(M_8,1-s,\overline{\psi}\right)=$$

$$\left[\frac{(p-1)\,p^{-1+2s}}{\prod_{k=1}^{3}\psi_{t_k}(p)}\right]\left[1-\left(\psi_{t_2}^{-1}(p)+\psi_{t_3}^{-1}(p)\right)p^{-1+s}+\psi_{t_2}^{-1}(p)\psi_{t_3}^{-1}(p)p^{-1+2s}\right]L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2,1-s,\overline{\psi}\right).$$

Therefore, we have the following equation:

$$\frac{\boldsymbol{L_{\Gamma_2}\left(M_8, s, \psi\right)}}{\boldsymbol{L_{\Gamma_2}\left(M_8, 1-s, \overline{\psi}\right)}} = \left[\boldsymbol{\psi_{t_1}^2(p)\psi_{t_2}^3(p)\psi_{t_3}^3(p)p^{3-6s}}\right] \frac{\boldsymbol{L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, s, \psi\right)}}{\boldsymbol{L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right)}}.$$

9) From [5, section 4], for $M_9 = \left\{(u_1, u_2, u_3) \in \mathbb{Z}_p^3 \colon\ u_1 - u_2 + u_3 \in p\mathbb{Z}_p\right\}$, we have:

a) $\mathrm{Aut}_{\Gamma_2}\left(M_9\right) = M_6^*$ and $\mu^*\left(M_6^*\right)^{-1} = (p-1)^2$.
b) $\{M_9 : \Gamma_2\} = (p, p^2, p^2)\, \widetilde{\Gamma}_2$.
c) $(M_9 : \Gamma_2) = p^2$.

Thus:

$$\boldsymbol{L_{\Gamma_2}\left(M_9, s, \psi\right) =}$$

$$\mu^*\left(\mathrm{Aut}_{\Gamma_2} M_9\right)^{-1}\left(\Gamma_2 : M_9\right)^{-s} \int\limits_{x \in \left(\mathbb{Q}_p^*\right)^3} \Phi_{\{M_9:\Gamma_2\}}\left(x\right)\psi(x)\left\|x\right\|_{\mathbb{Q}_p^3}^s d^*x =$$

$$(p-1)^2 p^{2s} \int\limits_{x \in \left(\mathbb{Q}_p^*\right)^3 \bigcap (p,p^2,p^2)\widetilde{\Gamma}_2} \psi(x)\left\|x\right\|_{\mathbb{Q}_p^3}^s d^*x =$$

$$(p-1)^2 \psi_{t_1}(p)\psi_{t_2}^2(p)\psi_{t_3}^2(p)p^{-3s}\left(\int\limits_{x \in \left(\mathbb{Q}_p^*\right)^3 \bigcap \widetilde{\Gamma}_2} \psi(x)\left\|x\right\|_{\mathbb{Q}_p^3}^s d^*x\right).$$

Therefore

$$\boldsymbol{L_{\Gamma_2}\left(M_9, s, \psi\right) = \left[(p-1)^2\psi_{t_1}(p)\psi_{t_2}^2(p)\psi_{t_3}^2(p)p^{-3s}\right] L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, s, \psi\right).}$$

Here are some examples:

$\boldsymbol{t_k}$	$\psi_{\mathbf{t_k}}(\mathbf{p})$	$\mathbf{L_{\Gamma_2}}\left(\mathbf{M_9}, \mathbf{s}, \psi\right)$
2	-1	$\frac{\left[-(p-1)^2 p^{-3s}\right]}{(1+p^{-s})^3}$
4	i	$\frac{\left[i(p-1)^2 p^{-3s}\right]}{(1-ip^{-s})^3}$.

We can see that

$$L_{\Gamma_2}\left(M_9, 1-s, \overline{\psi}\right) = \left[(p-1)^2 \psi_{t_1}^{-1}(p)\psi_{t_2}^{-2}(p)\psi_{t_3}^{-2}(p)p^{-3+3s}\right] L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right).$$

Therefore, we have the following equation:

$$\boldsymbol{\frac{L_{\Gamma_2}\left(M_9, s, \psi\right)}{L_{\Gamma_2}\left(M_9, 1-s, \overline{\psi}\right)} = \left[\psi_{t_1}^2(p)\psi_{t_2}^4(p)\psi_{t_3}^4(p)p^{3-6s}\right] \frac{L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, s, \psi\right)}{L_{\widetilde{\Gamma}_2}\left(\widetilde{\Gamma}_2, 1-s, \overline{\psi}\right)}.}$$

(3.4) Observation.Let $\Gamma_n = B_p\left(C_{p^n}\right)$ be the Burnside ring of the cyclic group of order p^n, for $1 < n \in \mathbb{N}$. In [5] we provide a resourceful method to obtain the ideals of finite index in Γ_n, and hence their isomorphism classes from the ideals of finite index in Γ_{n-1}, whereupon we finally obtain the corresponding $L-$ **functions**.

4. FUNCTIONAL EQUATION FOR $\mathbf{L_{B_p(G)}}\left(\mathbf{M}, \mathbf{s}, \psi\right)$.

(4.1) Notation.Let G be a finite group. We denote by $\Gamma = B_p(G)$ and by $\widetilde{\Gamma} = \prod_{H \in \mathcal{C}(G)} \mathbb{Z}_p$ its maximal order. Let M and N be representatives in the isomorphism classes of the fractional ideals of finite index in Γ, and let

$\{M : \Gamma\}$ be the conductor of M in Γ. Now we take A^* be the unit group of $A = \prod_{H\in\mathcal{C}(G)} \mathbb{Q}_p$. Let $t : A \to \mathbb{Q}_p$ be the map such that

$$t\left((y_H)_{H\in\mathcal{C}(G)}\right) = \sum_{H\in\mathcal{C}(G)} y_H,$$

for each $(y_H)_{H\in\mathcal{C}(G)} \in A$. We denote by

$$\overline{M} = \{a \in A : \quad t(xa) \in \mathbb{Z}_p \quad \forall \quad x \in \{M : \Gamma\}\}.$$

We choose a continuous additive character $\chi : \mathbb{Q}_p \to S^1$ on $\mathbb{Q}_p$, defined as

$$\chi(c) = \exp(2\pi i c_0),$$

where c_0 is the principal part of $c \in \mathbb{Q}_p$. We can observe that χ is trivial on $\mathbb{Z}_p$ but not on p^{-1}.

(4.2) Lemma.

Let $\Psi = \Phi_{\widetilde{\Gamma}}$ be the characteristic functions in A of $\widetilde{\Gamma}$. We have that the Fourier transform of Ψ is given by

$$\widehat{\Psi}(x) = \left(\widetilde{\Gamma} : \Gamma\right) \Psi(x),$$

for all $x \in A$.

.We consider the Fourier transform of Ψ, given by

$$\widehat{\Psi}(y) = \int_{x\in A} \Psi(x)\chi(t(xy))\, dx =$$

$$\int_{x\in\widetilde{\Gamma}} \chi(t(xy))\, dx,$$

for all $y \in A$.

We note that if $y \in \widetilde{\Gamma}$, then $xy \in \widetilde{\Gamma}$ for all $x \in \widetilde{\Gamma}$, where we can see that $\widehat{\Psi}(y) = \int_{x\in\widetilde{\Gamma}} dx$ for all $y \in \widetilde{\Gamma}$. We can consider $\int_{x\in\Gamma} dx = 1$, and thus

$$\int_{x\in\widetilde{\Gamma}} dx = \left(\widetilde{\Gamma} : \Gamma\right),$$

obtaining that $\widehat{\Psi}(y) = \left(\widetilde{\Gamma} : \Gamma\right)$ for all $y \in \widetilde{\Gamma}$.

On the other hand, let $y = (y_H)_{H\in\mathcal{C}(G)}$, such that $y \in A\diagdown\widetilde{\Gamma}$. We can consider $H \in \mathcal{C}(G)$, such that $y_H = p^{-n}x_H$ for some $x_H \in \mathbb{Z}_p$, and $0 < n \in \mathbb{Z}$. We consider the element $a = (0, ...0, 1, 0, ..., 0) \in A$ with coordinates zero and one in the H-th coordinate. We obtain $t(ay) = y_H$, and since χ is not trivial on p^{-1}, thus:

$$\chi\left(t\left(ay\right)\right) \neq 1.$$

Therefore, with the change of variable $x \to a + x$, we obtain

$$\widehat{\Psi}(y) = \int\limits_{x\in\widetilde{\Gamma}} \chi\left(t\left(ay\right) + t\left(xy\right)\right) dx = \int\limits_{x\in\widetilde{\Gamma}} \chi\left(t\left(ay\right)\right)\chi\left(t\left(xy\right)\right) dx =$$

$$\chi\left(t\left(ay\right)\right) \int\limits_{x\in\widetilde{\Gamma}} \chi\left(t\left(xy\right)\right) dx,$$

hence, $\widehat{\Psi}(y) = \chi\left(t\left(ay\right)\right)\widehat{\Psi}(y)$ and since $\chi\left(t\left(ay\right)\right) \neq 1$, we obtain $\widehat{\Psi}(y) = 0$ for all $y \in A\diagdown\widetilde{\Gamma}$, concluding that:

$$\widehat{\Psi}(x) = \left(\widetilde{\Gamma} : \Gamma\right)\Psi(x),$$

for all $x \in A$.

(4.3) Theorem.

From Notation (4.1), if M and N satisfy the condition

$$(*)\ \ \overline{M} = \alpha\left\{N : \Gamma\right\}\ \text{ for some } \alpha \in A^*,$$

then:

$$\boldsymbol{\frac{L_{\Gamma}\left(M, s, \psi\right)}{L_{\Gamma}\left(N, 1-s, \overline{\psi}\right)} = \left[\frac{\mu^*\left(\mathrm{Aut}_{\Gamma}N\right)\left\|\alpha\right\|^{1-s}\left(\Gamma : N\right)^{1-s}\overline{\psi}\left(\alpha\right)}{\mu^*\left(\mathrm{Aut}_{\Gamma}M\right)\left(\widetilde{\Gamma} : \left\{M : \Gamma\right\}\right)\left(\Gamma : M\right)^{s}}\right]\frac{L_{\widetilde{\Gamma}}\left(\widetilde{\Gamma}, s, \psi\right)}{L_{\widetilde{\Gamma}}\left(\widetilde{\Gamma}, 1-s, \overline{\psi}\right)}}.$$

.Let $\psi := A^* \to S^1$ be a continuous character. We choose a Haar measure dx on A and d^*x on the unit group A^*. For measurable sets $E \subseteq A$, $E' \subseteq A^*$, it will be convenient to write

$$\mu(E) = \int\limits_{E} dx, \quad \text{and} \quad \mu^*(E') = \int\limits_{E'} d^*x.$$

For this proof we will consider the functional equation given in (2.7) for the particular cases of $\Psi = \Phi_{\widetilde{\Gamma}}$ and $\Phi = \Phi_{\{M:\Gamma\}}$ the characteristic functions in A of $\widetilde{\Gamma}$ and the conductor of M on Γ, respectively.

(i)Let $\Psi(x) = \Phi_{\widetilde{\Gamma}}(x)$, for every $x \in A$. By definition, we have:

$$(1) \qquad Z(\Psi, s, \psi) = \int\limits_{x\in A^*} \Phi_{\widetilde{\Gamma}}(x)\psi(x)\,\|x\|_A^s\, d^*x.$$

We know that:

$$(2) \qquad L_{\widetilde{\Gamma}}\left(\widetilde{\Gamma}, s, \psi\right) = \mu^*\left(\widetilde{\Gamma}^*\right)^{-1} \int\limits_{x\in A^*} \Phi_{\widetilde{\Gamma}}(x)\psi(x)\,\|x\|_A^s\, d^*x.$$

We can consider $\mu^*\left(\widetilde{\Gamma}^*\right) = 1$, and then $\mu^*\left(\Gamma^*\right)^{-1} = \left(\widetilde{\Gamma}^* : \Gamma^*\right)$.

From (1) and (2), we obtain

$$(3) \qquad Z(\Psi, s, \psi) = L_{\widetilde{\Gamma}}\left(\widetilde{\Gamma}, s, \psi\right).$$

Now, we consider the Fourier transform of Ψ, given by:

$$\widehat{\Psi}(y) = \int\limits_{x\in A} \Psi(x)\chi\left(t\left(xy\right)\right) dx,$$

for all $y \in A$, and then

$$(4) \qquad \widehat{\Psi}(y) = \int\limits_{x\in\widetilde{\Gamma}} \chi\left(t\left(xy\right)\right) dx,$$

for all $y \in A$.

From lemma (4.2), we have that:

$$(5) \qquad \widehat{\Psi}(x) = \left(\widetilde{\Gamma} : \Gamma\right)\Psi(x).$$

Therefore, we have

$$Z(\widehat{\Psi}, s, \psi) = \left(\widetilde{\Gamma} : \Gamma\right) \int\limits_{x\in A^*} \Psi(x)\psi(x)\,\|x\|_A^s\, d^*x.$$

Thus from (2) , we have

$$Z(\widehat{\Psi}, s, \psi) = \left(\widetilde{\Gamma} : \Gamma\right) L_{\widetilde{\Gamma}}\left(\widetilde{\Gamma}, s, \psi\right)$$

whence

$$(6) \qquad Z(\widehat{\Psi}, 1-s, \overline{\psi}) = \left(\widetilde{\Gamma} : \Gamma\right) L_{\widetilde{\Gamma}}\left(\widetilde{\Gamma}, 1-s, \overline{\psi}\right).$$

Finally, from (3) and (6), we obtain

$$\textbf{(I)} \qquad \frac{\boldsymbol{Z(\Psi, s, \psi)}}{\boldsymbol{Z(\widehat{\Psi}, 1-s, \overline{\psi})}} = \left(\widetilde{\boldsymbol{\Gamma}} : \boldsymbol{\Gamma}\right)^{-1} \frac{\boldsymbol{L_{\widetilde{\Gamma}}\left(\widetilde{\Gamma}, s, \psi\right)}}{\boldsymbol{L_{\widetilde{\Gamma}}\left(\widetilde{\Gamma}, 1-s, \overline{\psi}\right)}}.$$

(ii)Let $\Phi(x) = \Phi_{\{M:\Gamma\}}(x)$, for every $x \in A$. By definition, we have

$$Z(\Phi, s, \psi) = \int\limits_{x \in A^*} \Phi_{\{M:\Gamma\}}(x)\psi(x) \left\|x\right\|_A^s d^*x.$$

We know from the definition that

$$(7) \quad L_\Gamma(M, s, \psi) = \mu^* \left(\mathrm{Aut}_\Gamma M\right)^{-1} (\Gamma : M)^{-s} \int\limits_{x \in A^*} \Phi_{\{M:\Gamma\}}(x)\psi(x) \left\|x\right\|_A^s d^*x,$$

whence

$$(8) \qquad Z(\Phi, s, \psi) = \mu^* \left(\mathrm{Aut}_\Gamma M\right) (\Gamma : M)^s L_\Gamma(M, s, \psi).$$

Now we consider the Fourier transform of Φ, given by

$$\widehat{\Phi}(y) = \int\limits_{x \in \{M:\Gamma\}} \chi\left(t\left(xy\right)\right) dx,$$

for all $y \in A$.

We have that $\mu\left(\{M : \Gamma\}\right) = \left(\{M : \Gamma\} : \Gamma\right)$. As in lemma (4.2), we can see that $\widehat{\Phi}(y) = \left(\{M : \Gamma\} : \Gamma\right) \Phi_{\overline{M}}(y)$, for all $y \in A$, whence we obtain, by hypothesis

$$\widehat{\Phi}(y) = \left(\{M : \Gamma\} : \Gamma\right) \Phi_{\alpha\{N:\Gamma\}}(y).$$

Thus

$$Z(\widehat{\Phi}, s, \psi) = \left(\{M : \Gamma\} : \Gamma\right) \int\limits_{x \in A^*} \Phi_{\{N:\Gamma\}}(\alpha^{-1}x)\psi(x) \left\|x\right\|_A^s d^*x =$$

$$\left(\{M : \Gamma\} : \Gamma\right) \left\|\alpha\right\|_A^s \psi(\alpha) \int\limits_{x \in A^*} \Phi_{\{N:\Gamma\}}(x)\psi(x) \left\|x\right\|_A^s d^*x.$$

Therefore, from (7), we have

$$Z(\widehat{\Phi}, s, \psi) = \left(\{M : \Gamma\} : \Gamma\right) \left\|\alpha\right\|_A^s \psi(\alpha)\mu^* \left(\mathrm{Aut}_\Gamma N\right) (\Gamma : N)^s L_\Gamma(N, s, \psi),$$

and then:

$$(9) \qquad Z(\widehat{\Phi}; 1-s, \overline{\psi}) =$$

$$\left(\{M : \Gamma\} : \Gamma\right) \left\|\alpha\right\|_A^{1-s} \overline{\psi}(\alpha)\mu^* \left(\mathrm{Aut}_\Gamma N\right) (\Gamma : N)^{1-s} L_\Gamma(N, 1-s, \overline{\psi}).$$

Finally, from (8) and (9), we have

$$\text{(II)}\quad \frac{\boldsymbol{Z(\Phi, s, \psi)}}{\boldsymbol{Z(\widehat{\Phi}; 1-s, \overline{\psi})}} =$$

$$\left[\frac{\boldsymbol{\mu^*}\left(\mathrm{Aut}_{\boldsymbol{\Gamma}}\boldsymbol{M}\right)\|\boldsymbol{\alpha}\|^{-1+s}\left(\boldsymbol{\Gamma}:\boldsymbol{M}\right)^{s}\left(\boldsymbol{\Gamma}:\boldsymbol{N}\right)^{-1+s}}{\boldsymbol{\mu^*}\left(\mathrm{Aut}_{\boldsymbol{\Gamma}}\boldsymbol{N}\right)\left(\{\boldsymbol{M}:\boldsymbol{\Gamma}\}:\boldsymbol{\Gamma}\right)\overline{\boldsymbol{\psi}}(\boldsymbol{\alpha})}\right]\frac{\boldsymbol{L_{\Gamma}(M, s, \psi)}}{\boldsymbol{L_{\Gamma}(N, 1-s, \overline{\psi})}}.$$

Now (I) and (II), plus the functional equation in (2.7), complete the proof of this theorem.

■

(4.4) Observation:

(a)Let $\Gamma_1 = B_p\left(C_p\right)$, and $\widetilde{\Gamma}_1 = \mathbb{Z}_p^2$. From [5, section 3] we can see that the only isomorphism classes of the fractional ideals of finite index of Γ_1, are $M_1 = \Gamma_1$, and its maximal order $M_2 = \widetilde{\Gamma}_1$, for which we have $\{\Gamma_1 : \Gamma_1\} = \Gamma_1$ and $\left\{\widetilde{\Gamma}_1 : \Gamma_1\right\} = (p, p)\,\widetilde{\Gamma}_1$. Hence:

1). We know that $\overline{M}_1 = \left\{(b_1, b_2) \in \mathbb{Q}_p^2 : \quad t\left(x\left(b_1, b_2\right)\right) \in \mathbb{Z}_p \quad \forall \quad x \in \Gamma_1\right\}$, which implies that

$$\overline{M}_1 = \left(p^{-1}, -p^{-1}\right)\Gamma_1 = \left(p^{-1}, -p^{-1}\right)\{\Gamma_1 : \Gamma_1\}.$$

2). We know that

$$\overline{M}_2 = \left\{(b_1, b_2) \in \mathbb{Q}_p^2 : \quad t\left(x\left(b_1, b_2\right)\right) \in \mathbb{Z}_p \quad \forall \quad x \in (p, p)\,\widetilde{\Gamma}_1\right\},$$

which implies that

$$\overline{M}_2 = \left(p^{-1}, p^{-1}\right)\widetilde{\Gamma}_1 = \left(p^{-2}, p^{-2}\right)\left\{\widetilde{\Gamma}_1 : \Gamma_1\right\}.$$

Therefore, in this case we can see that the condition $(*)$ of the previous theorem is satisfied for every isomorphism class of fractional ideals of Γ_1. Moreover, we can see that in this case, the functional equations which follow the theorem (4.3) are those obtained in (3.2).

(b) Let $\Gamma_2 = B_p\left(C_{p^2}\right)$, and $\widetilde{\Gamma}_1 = \mathbb{Z}_p^3$. From [5, section 4] we can see that the only isomorphism classes of the fractional ideals of finite index of Γ_2, are:

$M_1 = \Gamma_2$,

$M_2 = \widetilde{\Gamma}_2$,
$M_3 = \mathbb{Z}_p \bigoplus \Gamma_1$,
$M_4 = \Gamma_1 \bigoplus \mathbb{Z}_p$,
$M_5 = \left\{(u_1, u_2, u_3) \in \mathbb{Z}_p^3\colon\ (u_3 - u_1) \in p\mathbb{Z}_p\right\}$,
$M_6 = \left\{(u_1, u_2, u_3) \in \mathbb{Z}_p^3\colon\ (u_2 - u_1) \in p\mathbb{Z}_p, (u_3 - u_1) \in p\mathbb{Z}_p\right\}$,
$M_7 = \left\{(u_1, u_2, u_3) \in \mathbb{Z}_p^3\colon\ (u_3 - u_2) \in p^2\mathbb{Z}_p\right\}$,
$M_8 = \left\{(u_1, u_2, u_3) \in \mathbb{Z}_p^3\colon\ pu_1 - u_2 + u_3 \in p^2\mathbb{Z}_p\right\}$ and
$M_9 = \left\{(u_1, u_2, u_3) \in \mathbb{Z}_p^3\colon :\ u_1 - u_2 + u_3 \in p\mathbb{Z}_p\right\}$ for which we have:

1). For M_i, where $i = 2, 4, 5$ and 9, we have that $\{M_i : \Gamma_2\} = (p, p^2, p^2)\, M_2$. Therefore,

$$\overline{M}_i = \left\{(b_1, b_2, b_3) \in \mathbb{Q}_p^3 : \quad t\left(x\left(b_1, b_2, b_3\right)\right) \in \mathbb{Z}_p \quad \forall \quad x \in \left(p, p^2, p^2\right) M_2\right\},$$

whence clearly

$$\overline{M}_i = \left(p^{-1}, p^{-2}, p^{-2}\right) M_2 = \left(p^{-2}, p^{-4}, p^{-4}\right) \{M_2 : \Gamma_2\}.$$

2). For M_j, where $j = 3, 6, 7$ and 8, we have that $\{M_j : \Gamma_2\} = (p, p, p)\, M_3$. Therefore,

$$\overline{M}_j = \left\{(b_1, b_2, b_3) \in \mathbb{Q}_p^3 : \quad t\left(x\left(b_1, b_2, b_3\right)\right) \in \mathbb{Z}_p \quad \forall \quad x \in (p, p, p)\, M_3\right\},$$

whence clearly

$$\overline{M}_j = \left(p^{-1}, p^{-2}, -p^{-2}\right) M_3 = \left(p^{-2}, p^{-3}, p^{-3}\right) \{M_3 : \Gamma_2\}.$$

3). For M_1, we have $\{M_1 : \Gamma_2\} = M_1$. Thus

$$\overline{M}_1 = \left\{(b_1, b_2, b_3) \in \mathbb{Q}_p^3 : \quad t\left(x\left(b_1, b_2, b_3\right)\right) \in \mathbb{Z}_p \quad \forall \quad x \in M_1\right\},$$

whence clearly

$$\overline{M}_1 = \left(p^{-1}, -p^{-2}, p^{-2}\right) M_8.$$

From the three points above, we can see that for M_k, where $k = 2, ..., 9$, the condition $(*)$ is fulfilled. Moreover, we can see that in this case the functional equations which follow the theorem (4.3) are those obtained in (3.3). On the other hand, the case M_1 does not satisfy the condition $(*)$, therefore, we deduce that this is a non-trivial condition, since it is not generally satisfied for all isomorphism classes of the fractional ideals of $\Gamma = B_p\left(G\right)$. However, if $\widetilde{\Gamma}$ is the maximal order of Γ, it is always a representative of the isomorphism classes of the fractional ideals of Γ for every finite group G, for which the condition $(*)$ is always fulfilled, as shown in the following proposition.

(4.5) PropositionLet G be a finite group, and let $\Gamma = B_p(G)$ be the Burnside ring of G. Let $\widetilde{\Gamma} = \prod_{H\in\mathcal{C}(G)} \mathbb{Z}_p$ be the maximal order of Γ and let $A = \prod_{H\in\mathcal{C}(G)} \mathbb{Q}_p$. Then

$$\widetilde{\widetilde{\Gamma}} = \beta^{-2}\left\{\widetilde{\Gamma} : \Gamma\right\},$$

for some $\beta \in A^*$.

. We observe that $\left\{\widetilde{\Gamma} : \Gamma\right\}$ is a $\widetilde{\Gamma}-$lattice, and thus $\left\{\widetilde{\Gamma} : \Gamma\right\} = \beta\widetilde{\Gamma}$, for some $\beta \in A^*$, whence

$$\widetilde{\widetilde{\Gamma}} = \left\{b \in A : \quad t(xb) \in \mathbb{Z}_p \quad \forall \quad x \in \beta\widetilde{\Gamma}\right\}.$$

Suppose that $b \in \beta^{-1}\widetilde{\Gamma}$, then we have $xb \in \widetilde{\Gamma}$ for all $x \in \beta\widetilde{\Gamma}$. Thus $t(xb) \in \mathbb{Z}_p$, hence $\beta^{-1}\widetilde{\Gamma} \subseteq \widetilde{\widetilde{\Gamma}}$. Now, suppose that $b \in A \setminus \beta^{-1}\widetilde{\Gamma}$. Then we have $\beta b \notin \widetilde{\Gamma}$, and thus, if $\beta b = (y_H)_{H\in\mathcal{C}(G)}$, then there is an isomorphism class $H_0 \in \mathcal{C}(G)$ such that $y_{H_0} = p^{-n}x_{H_0}$ for $0 < n \in \mathbb{Z}$, and $x_{H_0} \in \mathbb{Z}_p$. Let $e_{H_0} \in \widetilde{\Gamma}$ be the element with coordinates zero and one in the $H_0 -$ th coordinate. We have $t(e_{H_0}\beta b) = p^{-n}x_{H_0} \notin \mathbb{Z}_p$, and thus we obtain

$$\widetilde{\widetilde{\Gamma}} = \beta^{-1}\widetilde{\Gamma} = \beta^{-2}\left\{\widetilde{\Gamma} : \Gamma\right\},$$

which is the conclusion of this proof.

■

(4.6) Example.As a particular case of the previous proposition, let Γ_n be the Burnside ring of the cyclic group of order p^n, where $2 < n \in \mathbb{Z}$ and let $\widetilde{\Gamma}_n$ be its maximal order. We have that

$$\Gamma_n = \left\{(u_1, \ldots, u_{n+1}) \in \mathbb{Z}_p^{(n+1)} : \quad (u_l - u_{l-1}) \in p^{l-1}\mathbb{Z}_p \text{ for } l = 2, \ldots, n+1\right\},$$

and $\widetilde{\Gamma}_n = \mathbb{Z}_p^{(n+1)}$. Let $b = (b_1, ..., b_{n+1}) \in \left\{\widetilde{\Gamma}_n : \Gamma_n\right\}$, and let $e_1, ..., e_{n+1}$ be the canonical basis of $\widetilde{\Gamma}_n$. Then we have the condition $e_i b \in \Gamma_n$, which implies that $b_i \in p^i\mathbb{Z}_p$ for $i = 1, ..., n$, and $b_{n+1} \in p^n\mathbb{Z}_p$, where it is clear that

$$\left\{\widetilde{\Gamma}_n : \Gamma_n\right\} = \left(p, p^2, ..., p^{n-1}, p^n, p^n\right)\widetilde{\Gamma}_n.$$

Therefore, from the proof of the previous proposition, we obtain

$$\widetilde{\widetilde{\Gamma}}_n = \left(p^{-2}, p^{-4}, ..., p^{-2(n-1)}, p^{-2n}, p^{-2n}\right)\left\{\widetilde{\Gamma}_n : \Gamma_n\right\},$$

and from the previous theorem we obtain the following functional equation:

$$\frac{L_{\Gamma_n}\left(\widetilde{\Gamma}_n, s, \psi\right)}{L_{\Gamma_n}\left(\widetilde{\Gamma}_n, 1-s, \overline{\psi}\right)} = \left[\frac{\|\alpha\|^{1-s}\left(\Gamma_n : \widetilde{\Gamma}_n\right)^{1-2s}\overline{\psi}\left(\alpha\right)}{\left(\widetilde{\Gamma}_n : \left\{\widetilde{\Gamma}_n : \Gamma_n\right\}\right)}\right] \frac{L_{\widetilde{\Gamma}_n}\left(\widetilde{\Gamma}_n, s, \psi\right)}{L_{\widetilde{\Gamma}_n}\left(\widetilde{\Gamma}_n, 1-s, \overline{\psi}\right)},$$

where

$$\alpha = \left(p^{-2}, p^{-4}, ..., p^{-2(n-1)}, p^{-2n}, p^{-2n}\right),$$

and

$$\left(\widetilde{\Gamma}_n : \Gamma_n\right) = \prod_{i=1}^{n} p^i.$$

Let $\psi = \left(\psi_{t_1}, ..., \psi_{t_{n+1}}\right) : \left(\mathbb{Q}_p^*\right)^{(n+1)} \to S^1$ be a continuous character of finite order, which is trivial on $\left(\mathbb{Z}_p^*\right)^{(n+1)}$, where

$$\psi_t : \mathbb{Q}_p^* \to S^1$$

is defined by $\psi_t\left(p\right) = \exp\left(\frac{2\pi i}{t}\right)$, for $0 < t \in \mathbb{Z}$, and

$$\psi\left(b_1, ..., b_{n+1}\right) = \prod_{i=1}^{n+1} \psi_{t_i}\left(b_i\right).$$

Therefore,

$$\boldsymbol{\frac{L_{\Gamma_n}\left(\widetilde{\Gamma}_n, s, \psi\right)}{L_{\Gamma_n}\left(\widetilde{\Gamma}_n, 1-s, \overline{\psi}\right)} = \left[\psi_{t_{n+1}}^{2n}\left(p\right) \prod_{k=1}^{n} \psi_{t_k}^{2k}\left(p\right)\right] \left[p^n\right]^{1-2s} \frac{L_{\widetilde{\Gamma}_n}\left(\widetilde{\Gamma}_n, s, \psi\right)}{L_{\widetilde{\Gamma}_n}\left(\widetilde{\Gamma}_n, 1-s, \overline{\psi}\right)}}.$$

Acknowledgment. I am grateful to Prof. Dr. Alberto Gerardo Raggi Cárdenas, for his many valuable suggestions.

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Received: July 5, 2014