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Soft BCL-Algebras

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Abstract

A BCL-algebra is a class of algebras of type (2, 0) was defined in 2011 by Y. Liu. In this paper, the notion of soft BCL-algebra is introduced. Then some binary operations between two soft BCL-algebras are studied. Moreover, we find relations between soft BCL-algebra and soft BCK/BCI/BCH/d –algebra.

Keywords: BCL/BCK/BCI/BCH/d–algebra, Soft BCL/BCK/BCI/BCH/d–algebra

1 Introduction

The aim of this paper is to study soft BCL-algebra and investigate relations between soft BCK/BCI/BCH/d–algebra. Molodtsov introduced, in [10], the basic notions of the theory of soft sets to deal with uncertainties when solving problems in practice as in environment, economics, social science and engineering. This theory is convenient and easy to apply as it is free from the difficulties that appear when using other mathematical tools as theory of fuzzy sets, theory of vague sets and theory of rough sets etc. (For more details we refer the reader to [10]).

Historically, two classes of abstract algebras: BCK and BCI-algebras has been introduced in 1966 ([3] and [4]). Then in 1983, the notion of BCK and BCI-algebras has been generalized by introducing the class of BCH-algebra. In 2011, Liu in his paper [9] has introduced the BCL-algebra which is more general class than BCK/BCI/BCH-algebra. Some properties of BCL-algebra are studied in [2]. Many authors applied the notion of soft set theory on several classes of algebra. For example, Jun in [5], introduced soft BCK/BCI-algebras and derived some of their properties. Also Kazanci et al. [7] introduced the concept of soft BCH-algebra and some of their properties and structural characteristics are studied. In this paper we will apply the notion of soft sets on BCL-algebra.

The paper is organized as follow. In Section 2, basic definitions of the related algebra is given and some binary operations on soft sets have been recalled. Section 3 of this paper introduces the notion of soft BCL-algebra and BCL-sublagebra. In the final section we construct new soft BCL-algebras from given ones. We conclude the section with Theorem 4.6, Theorem 4.7, Theorem 4.8 and Theorem 4.10 that relates soft BCL-algebra and soft BCK/BCI-algebra, BCH-algebra and d-algebra. We illustrate with examples throughout the paper.

2 Preliminaries

We start by defining a BCL-algebra. We refer the reader to [8], [5] and [7] for the definitions of d-algebra, BCK/BCI-algebra and BCH-algebra respectively.

Definition 2.1: [9, Definition 2.1] An algebra $(X; *, 0)$ of type $(2, 0)$ is a BCL-algebra if it satisfies the following conditions for any $x, y, z \in X$:

1) BCL-1: $x * x = 0$;
2) BCL-2: $x * y = 0$ and $y * x = 0$ imply $x = y$;
3) BCL-3: $\big((x * y) * z\big) * \big((x * z) * y\big) * \big((z * y) * x\big) = 0$.

In what follows, U is an initial universe set and E is a set of parameters. The power set of U is denoted by $\mathcal{P}(U)$ and A is a subset of E.

Definition 2.2: [10] A pair (F, E) is called a soft set over U if F is a mapping of E into the set of all subsets of the set U.

Definition 2.3: [5, Definition 3.3] Let (F, A) and (G, B) be two soft sets over a common universe U. The union of two soft sets (F, A) and (G, B) over the universe U is the soft set (H, C) where

1. $C = A \cup B$,
2. for all $c \in C$,

$$H(c) = \begin{cases} F(c) & if \quad c \in A \setminus B \\ G(c) & if \quad c \in B \setminus A \\ F(c) \cup G(c) & if \quad c \in A \cap B. \end{cases}$$

We write $(H, C) = (F, A) \,\tilde{\cup}\, (G, B)$.

Definition 2.4: [5, Definition 3.2] Let (F,A) and (G,B) be two soft sets over a common universe U. The intersection of (F,A) and (G,B) is the soft set (H,C) where

1. $C = A \cap B$,
2. for all $c \in C$, H(c) = F(c) or G(c).

We write $(H,C) = (F,A) \tilde{\cap} (G,B)$.

Remark 2.1: Pie and Miao pointed out in [11] that generally F(c) or G(c) may not be identical. To avoid degenerate case Ahmad and Kharal in [1] proposed that $A \cap B \neq \phi$.

Definition 2.5: [7, Definition 2.9] Let (F,A) and (G,B) be two soft sets over a common universe U. The Cartesian product of (F,A) and (G,B) is the soft set (H,C) where $H: C \rightarrow \mathcal{P}(U \times U)$ and

1. $C = A \times B$,
2. for all $(x,y) \in C$, $H(x,y) = F(x) \times G(y)$.

We write $(H,C) = (F,A) \times (G,B)$.

Definition 2.6: [6, Definition 4.3] Let (F,A) be a soft set over X. Then (F,A) is called a soft d-algebra over X if (F(x); $*$, 0) is a d-algebra for all x ∈ A.

We refer the reader to [5] and [7] for definitions for soft BCK/BCI-algebra and soft BCH-algebra respectively.

3 Soft BCL-algebra

In this section we introduce the notion of BCL-subalgebra and soft BCL-algebra. We will illustrate the definitions with examples.

Definition 3.1: A non-empty subset S of a BCL-algebra X is called a BCL-subalgebra of X if $x * y \in S$ for all $x, y \in S$.

Definition 3.2: If X is a BCL-algebra and $A \neq \phi$. A set valued function $F: A \rightarrow \mathcal{P}(X)$ can be defined by $F(x) = \{ y \in X \mid xRy\}$ for all $x \in A$, where R is an arbitrary binary operation from A to X. The pair (F,A) is then a soft set over X.

Definition 3.3: Let (F,A) be a soft set over X. Then (F,A) is called a soft BCL-algebra over X if $F(x)$ is a BCL-subalgebra of X for all $x \in A$.

Example 3.1: Let $X := \{0,1,a,b\}$ be a BCL-algebra with the following Cayley table:

$*$	0	1	a	b
0	0	0	0	0
1	1	0	b	1
a	a	b	0	a
b	b	a	0	0

Let (F,A) be a soft set over X, where $A = X$ and $F: A \to \mathcal{P}(X)$ is a set valued function defined by

$F(x) = \{y \in X \mid xRy \Leftrightarrow y = x^n, n \in \mathbb{N}\}$ for all $x \in A$.

Then $F(0) = \{0\}, F(1) = \{1,0\}, F(a) = \{a,0\}, F(b) = \{b,0\}$ which are BCL-subalgebras of X using Definition 3.1. Hence, by Definition 3.3**,** (F,A) is a soft BCL-algebra over X.

The next example shows that there exist set-valued functions $F: A \to \mathcal{P}(X)$ where the soft set (F,A) is not a soft BCL-algebra over X.

Example 3.2: Consider the BCL-algebra in Example 3.1 with a set valued function defined by

$F(x) = \{y \in X \mid xRy \Leftrightarrow y * x \in \{1,b\}\}$ for all $x \in A$.

We have, $F(0) = \{1,b\}, F(1) = \{a\}, F(a) = \{1\}, F(b) = \{1\}$ are not BCL-subalgebras of X. Therefore, (F,A) is not a soft BCL-algebra over X.

4 Results on soft BCL-algebras

In this section our results are presented. We start with the following Lemma.

Lemma 4.1: If (F,A) is a soft BCL-algebra over X. If B is a subset of A, then $(F_{|B}, B)$ is a soft BCL-algebra over X.

Proof: Straightforward

Remark 4.1: If (F,A) is not a soft BCL-algebra over X then there exists a subset B of A, such that $(F_{|B}, B)$ is a soft BCL-algebra over X as shown in the next example.

Example 4.1: Consider the BCL-algebra X in Example 3.1.
Let (F,A) be a soft set over X, where $A = X$ and $F: A \to \mathcal{P}(X)$ is a set valued function defined by

$F(x) = \{1\} \cup \{y \in X \mid xRy \Leftrightarrow y = x^n, n \in \mathbb{N}\}$ for all $x \in A$.
Thus, $F(0) = \{1, 0\}, F(1) = \{1, 0\}, F(a) = \{1, a, 0\}, F(b) = \{1, b, 0\}$. Note that $F(a)$ and $F(b)$ are not BCL-subalgebras of X. If we take $B = \{0, 1\} \subset X$ and

$F_{|B}(x) = \{1\} \cup \{y \in X \mid xRy \Leftrightarrow y = x^n, n \in \mathbb{N}\}$ for all $x \in B$,
we have $(F_{|B}, B)$ a soft BCL-algebra of X.

Definition 4.1: Let (F, A) be a soft BCL-algebra over X.

i) (F, A) is called the trivial soft BCL-algebra over X if $F(x) = \{0\}$ for all $x \in A$.
ii) (F, A) is called the whole soft BCL-algebra over X if $F(x) = X$ for all $x \in A$.

Example 4.2: Consider the BCL-algebra $(X; *, 0)$ given in [8, p 298] with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	3	1
2	2	3	0	2
3	3	0	0	0

If we take $A = \{1,2\}$ and $F(x) = \{y \in X \mid y * x \in \{0,3\}\}$ for all $x \in A$, then $F(1) = \{0, 1, 2, 3\}$ and $F(2) = \{0, 1, 2, 3\}$. We can see that $F(x) = X$ for all $x \in A$. Thus (F, A) is a whole soft BCL-algebra over X.

If we take $F(x) = \{y \in X \mid y * x = y\}$ for all $x \in A$, then $F(1) = \{0\}$ and $F(2) = \{0\}$. Thus (F, A) is a trivial soft BCL-algebra over X.

To construct new soft BCL-algebras from given ones, the operations on soft sets that have been defined in Section 1 is applied next. Note that the proof of Theorem 4.1 and Theorem 4.2 are similar to the proof of Theorem 4.9 and Corollary 4.8 in [5] and is written here for completeness.

Theorem 4.1: Let (F, A) and (G, B) be two soft BCL-algebras over X. If $A \cap B = \phi$, then the union $(H, C) = (F, A) \tilde{\cup} (G, B)$ is a soft BCL-algebra over X.

Proof: From Definition 2.3 with $A \cap B = \phi$, we have for all $c \in C$,

$$H(c) = \begin{cases} F(c) & if \quad c \in A\backslash B \\ G(c) & if \quad c \in B\backslash A. \end{cases}$$

If $c \in A\backslash B$ then $H(c) = F(c)$ is a BCL-subalgebra of X. Similarly, if $c \in B\backslash A$ then $H(c) = G(c)$ is a BCL-subalgebra of X. Hence $(H, C) = (F, A) \tilde{\cup} (G, B)$ is a soft BCL-algebra over X. Thus the union of two soft BCL-algebras is a soft BCL-algebra.

Example 4.3: Let $X \coloneqq \{0, 1, 2, 3\}$ be a BCL-algebra with the following Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	3	1
2	2	3	0	2
3	3	2	0	0

Let (F, A) and (G, B) be two soft sets where $A \coloneqq \{0, 1, 2\}$ and $B \coloneqq \{3\}$. Define $F: A \to \mathcal{P}(X)$ such that $F(a) = \{y \in X \mid y * a \in \{0, 1\}\}$ and $G: B \to \mathcal{P}(X)$ such that $G(b) = \{y \in X \mid y * b \in \{2\}\}$. Note that $A \cap B = \phi$ and $C = A \cup B = X$. We have, $F(0) = F(1) = \{0, 1\}, F(2) = \{0, 2, 3\}$ and $G(3) = \{2\}$. Thus, $H(0) = H(1) = \{0, 1\}, H(2) = \{0, 2, 3\}, H(3) = \{2\}$ which are BCL-subalgebras of X. Hence, (H, C) is a soft BCL-algebra over X.

Remark 4.2: The condition $A \cap B = \phi$ is important as if $A \cap B \neq \phi$ then the theorem does not apply. For example, if $A \coloneqq \{0, 1, 2\}$ and $B \coloneqq \{0, 3\}$. Then $H(0) = F(0) \cup G(0) = \{0, 1, 2\}$ which is not a BCL-subalgebra. Therefore, (H, C) is not a soft BCL-algebra over X in this case.

Theorem 4.2: Let (F, A) and (G, B) be two soft BCL-algebras over X. If $A \cap B \neq \phi$, then the intersection $(H, C) = (F, A) \tilde{\cap} (G, B)$ is a soft BCL-algebra over X.

Proof: Recall Definition 2.4 in which we wrote $(H, C) = (F, A) \tilde{\cap} (G, B)$ where $C = A \cap B$ and H(c) = F(c) or G(c) for all $c \in C$. Note that $H: C \to \mathcal{P}(X)$ is a mapping and so (H, C) is a soft set over X. We have, $H(c) = F(c)$ is a BCL-subalgebra of X or $H(c) = G(c)$ is a BCL-subalgebra of X. Hence, $(H, C) = (F, A) \tilde{\cap} (G, B)$ is a soft BCL-algebra over X. Therefore, the intersection of two soft BCL-algebras is a soft BCL-algebra.

Example 4.4: Consider the algebra in Example 4.3 with $A \coloneqq \{0, 1, 2\}$ and $B \coloneqq \{0, 3\}$. Then $H(0) = F(0) = \{0, 1\}$ or $G(0) = \{2\}$. Note that both are BCL-subalgebras of X. Hence, (H, C) is a soft BCL-algebra over X.

Theorem 4.3: Let (F, A) and (G, B) be two soft BCL-algebras over X. Then the cartisian product $(H, C) = (F, A) \times (G, B)$ is a soft BCL-algebra over $X \times X$.

Proof: From Definition 2.5 we can write $(H, C) = (F, A) \times (G, B)$ where $C = A \times B$ and for all $(x, y) \in C$, $H(x, y) = F(x) \times G(y)$. As (F, A) is a soft BCL-algebra over X, we have that $F(x)$ is a BCL-subalgebra of X. Similarly , $G(y)$ is a BCL-subalgebra of X. Therefore, $(H, C) = (F, A) \times (G, B)$ is a soft BCL-algebra over $X \times X$.

Example 4.5: Let $A \coloneqq \{0, 1, 2\}$ and $B \coloneqq \{0, a, b\}$ be two BCL-algebras over X and Y, respectively with the following Cayley table:

*	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

*	0	a	b
0	0	0	0
a	a	0	0
b	b	b	0

Then $A \times B = \{(0,0), (0, a), (0, b), (1,0), (1, a), (1, b), (2,0), (2, a), (2, b)\}$ is a BCL-algebra with the following Cayley table.

*	(0,0)	$(0,a)$	$(0,b)$	(1,0)	$(1,a)$	$(1,b)$	(2,0)	$(2,a)$	$(2,b)$
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
$(0,a)$	$(0,a)$	(0,0)	(0,0)	$(0,a)$	(0,0)	(0,0)	$(0,a)$	(0,0)	(0,0)
$(0,b)$	$(0,b)$	$(0,b)$	(0,0)	$(0,b)$	$(0,b)$	(0,0)	$(0,b)$	$(0,b)$	(0,0)
(1,0)	(1,0)	(1,0)	(1,0)	(0,0)	(0,0)	(0,0)	(1,0)	(1,0)	(1,0)
$(1,a)$	$(1,a)$	(1,0)	(1,0)	$(0,a)$	(0,0)	(0,0)	$(1,a)$	(1,0)	(1,0)
$(1,b)$	$(1,b)$	$(1,b)$	(1,0)	$(0,b)$	$(0,b)$	(0,0)	$(1,b)$	$(1,b)$	(1,0)
(2,0)	(2,0)	(2,0)	(2,0)	(2,0)	(2,0)	(2,0)	(0,0)	(0,0)	(0,0)
$(2,a)$	$(2,a)$	(2,0)	(2,0)	$(2,a)$	(2,0)	(2,0)	$(0,a)$	(0,0)	(0,0)
$(2,b)$	$(2,b)$	$(2,b)$	(2,0)	$(2,b)$	$(2,b)$	(2,0)	$(0,b)$	$(0,b)$	(0,0)

Let (F, A) and (G, B) be two soft BCL-algebras over X and Y, respectively. Define the set valued functions $F: A \to \mathcal{P}(X)$ by

$$F(x) = \begin{cases} \{0, 1\} & if \quad x \in \{0, 1\} \\ \{0, 2\} & if \quad x = 2, \end{cases}$$

and $G: B \to \mathcal{P}(X)$ by

$$G(y) = \begin{cases} \{0\} & if \quad y = 0 \\ \{0, a\} & if \quad y = a \\ \{0, b\} & if \quad y = b. \end{cases}$$

$H(x,y) = F(x) \times G(y), \forall\ (x,y) \in A \times B.$ Then we have,
$H(0,0) = F(0) \times G(0) = \{0\ 1\} \times \{0\} = \{(0,0),(0,1)\}.$ Similarly,
$H(0,a) = \{(0,0),(0,a),(1,0),(1,a)\},\ H(0,b) = \{(0,0),(0,b),(1,0),(1,b)\},$
$H(1,0) = \{(0,0),(0,1)\},\ H(1,a) = \{(0,0),(0,a),(1,0),(1,a)\},$
$H(1,b) = \{(0,0),(0,b),(1,0),(1,b)\},\ H(2,0) = \{(0,0),(2,0)\},$
$H(2,a) = \{(0,0),(0,a),(2,0),(2,a)\}$ and $H(2,b) = \{(0,0),(0,b),(2,0),(2,b)\}.$
It can be checked easily that for all $(x,y) \in A \times B$, $H(x,y)$ are BCL-subalgebras of X. Thus (H,C) is a soft BCL-algebra over $X \times Y$.

In the next part, Theorem 4.6, Theorem 4.7, Theorem 4.8 and Theorem 4.10 shows that a soft BCL-algebra could be soft BCL/ BCK/BCI/BCH/d –algebra and vice versa. We will recall some related theorems.

Theorem 4.4: [2, Theorem 3.1] A d-algebra X satisfying $(x * y) * z = (x * z) * y$ for any $x, y, z \in X$ is a BCL-algebra

Theorem 4.5: [2, Theorem 3.2] A BCL-algebra X satisfying $0 * x = 0$ for any $x \in X$ is a d-algebra.

Theorem 4.6: Let (X; ∗, 0) is a BCL-algebra with the condition $0 * x = 0$. If (F,A) is a soft BCL-algebra over X then (F,A) is a soft d-algebra over X.

Proof: Straightforward from Definition 3.3, Theorem 4.5 and Definition 2.6.

Theorem 4.7: If (X; ∗, 0) is a d-algebra with the condition $(x * y) * z = (x * z) * y$ and (F,A) is a soft d-algebra over X. Then (F,A) is a soft BCL-algebra over X.

Proof: Straightforward from Definition 2.6, Theorem 4.4 and Definition 3.3.

From [9, Theorem 2.1], we know that any BCK/BCI/BCH-algebra is a BCL-algebra. Thus we have the following theorem (the proves are direct).

Theorem 4.8: If (X; ∗, 0) is a BCK/BCI/BCH-algebra and (F,A) is a soft BCK/BCI/BCH-algebra over X. Then (F,A) is a soft BCL-algebra.

Theorem 4.9: (See [2,Theorem 3.5]) Let (X;∗ ,0) be a BCL-algebra. If $0 * x = 0$ and $x * y = x * z$ for any $x, y, z \in X$, then

1) the BCL-algebra is a BCK-algebra;
2) the BCL-algebra is a BCI-algebra.

Theorem 4.10: If $(X; *, 0)$ is a BCL-algebra and (F,A) is a soft BCL-algebra over X. Then (F,A) is a soft BCK/BCI-algebra.

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