International Journal of Algebra, Vol. 7, 2013, no. 12, 549 - 557
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ija.2013.3548

Serre Subcategory in Abelian Category

R. Rasuli

Mathematics Department
Faculty of Science Payame Noor University(PNU)
Tehran, Iran
rasulirasul@yahoo.com

Abstract. Let $\mathcal{C}$ and $\mathcal{S}$ be an abelian and serre subcategory respectively. In this paper we define the exact sequence, dimension, projective and injective object respect to $\mathcal{S}$. Finally we prove some results about them.

Mathematics Subject Classification: 13D45

1. Introduction

throughout this paper $\mathcal{C}$ will be an abelian category. Always $\mathcal{S}$ stands for a serre subcategory of $\mathcal{C}.\mathcal{S}$ is said to be a serre class (or serre subcategory, if it is closed under taking subobjects, quotients and extensions. we define $\mathcal{S}$-exact sequence, $\mathcal{S}-\mathrm{pd}(M)$,$\mathcal{S}-ind(M)$ and $\mathcal{S}-\dim(M)$. Note that the following subcategories are examples of serre subcategory of the category of R-modules: finite R-modules; coatomic R-modules [2];minimax R-modules [1]; and trivially the zero R-modules.

2. The main results

We begin this section with the following definition:

Definition 2.1. Let $\varphi : A \to B$ and $\psi : B \to C$ be morphism in $\mathcal{C}$. The sequence $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ is called $\mathcal{S}$-exact if $\ker\psi/Im\varphi \in \mathcal{S}$. Let $\mathcal{S}$ be a subcategory of the abelian category $\mathcal{C}$ then $\mathcal{S}$ is said to be a serre class, if for any $\mathcal{S}$-exact sequence

$$I \to L \to M \to N \to T$$

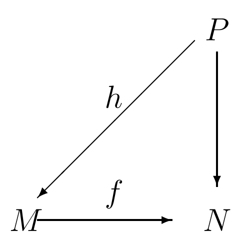
with Intitial object I and terminal object T in $\mathcal{C}$,M belongs to $\mathcal{S}$ if and only if I,L,N,T belong to $\mathcal{S}$. Always, we assume that Abelain category $\mathcal{C}$ has the null object 0.

Example 2.2. Let M be an object in $\mathcal{C}$, then the sequence $0 \to M \xrightarrow{1_M} M$ and $M \xrightarrow{1_M} M \to 0$ are $\mathcal{S}$-exact sequence.

Definition 2.3. Let $f : M \to N$ be a morphism in $\mathcal{C}$,f is called $\mathcal{S}$-monic if $Kerf \in \mathcal{S}$, and it is called $\mathcal{S}$-epic if $Cokerf \in \mathcal{S}$.

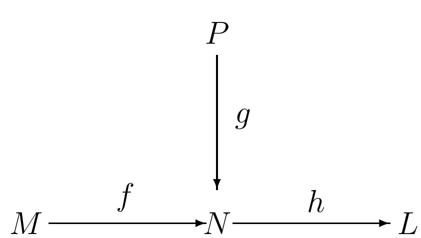
Example 2.4. (a) The sequence $0 \to M \xrightarrow{\varphi} N$ is $\mathcal{S}$-exact if and only if φ is $\mathcal{S}$-monic.
(b)The sequence $M \xrightarrow{\varphi} N \to 0$ is $\mathcal{S}$-exact if and only if φ is $\mathcal{S}$-epic.
(c) If $f : M \to N$ is $\mathcal{S}$-monic ($\mathcal{S}$-epic)then $0 \to M \xrightarrow{f} N \to Cokerf \to 0$ $(0 \to Kerf \to M \xrightarrow{f} N \to 0$) will be $\mathcal{S}$-exact.
(d) The sequence $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ is $\mathcal{S}$-exact if and only if φ is $\mathcal{S}$-monic,ψ is $\mathcal{S}$-epic and $\ker\psi/Im\varphi \in \mathcal{S}$. We call this sequence the short $\mathcal{S}$-exact sequence. For example if L be subobject of M, then the sequence $0 \to L \to M \to M/L \to 0$ will be short $\mathcal{S}$-exact sequence.

Definition 2.5. an object P of abelian category $\mathcal{C}$ is said $\mathcal{S}$-projective object if whenever there is a $\mathcal{S}$-epic morphism $f : M \to N$, and an arbitrary morphism $g : P \to N$, then there is a lifting $h : P \to M$ so that $foh = g$. Pictorially we have



such that f is $\mathcal{S}$-epic. The dually holds for $\mathcal{S}$-injective object.

Theorem 2.6. *Consider the diagram*



with the arow is $\mathcal{S}$-exact,P is $\mathcal{S}$-projective object. If $hog = 0$, then there exists morphism $\theta : P \to \ M$ such that $fo\theta = g$.

Proof. Let $X = Kerg/Imf$ and we have the diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle g} & & \\ M & \xrightarrow{f} & X & \xrightarrow{h} & 0 \end{array}$$

with arow is $\mathcal{S}$-exact. Since P is $\mathcal{S}$-projective then we have $\theta : P \to M$ such that $fo\theta = g$. □

Corollary 2.7. *Let P be $\mathcal{S}$-projective object and diagram*

$$\begin{array}{ccccc} P & \xrightarrow{f} & M & \xrightarrow{g} & N \\ & & \downarrow{\scriptstyle h} & & \\ A & \xrightarrow{\beta} & B & \xrightarrow{\alpha} & C \end{array}$$

be commutative. If arows are are $\mathcal{S}$-exact then we have $\theta : P \to A$ such that $\beta\theta = hf$.

Proof. Since P is $\mathcal{S}$-projective then there exists

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle \theta}\swarrow & \downarrow{\scriptstyle hf} & & \\ A & \xrightarrow{\beta} & B & \xrightarrow{\alpha} & C \end{array}$$

such that $\beta\theta = hf$. □

Definition 2.8. In an abelian category $\mathcal{C}$, a chain complex is a sequence $\mathbf{C} = \ldots \to C_{n+1} \overset{\partial_{n+1}}{\to} C_n \overset{\partial_n}{\to} C_{n-1} \to \ldots$, of composable arrow, with $\partial n \partial_{n+1} = 0$ for all n. The n-Th homology object is $H_nC = Ker(\partial_n : C_n \to C_{n-1})/Im(\partial_{n+1} \to C_n) = Z_n(C)/B_n(C) = Z_n/B_n$ and thus $H_nC = Z_n/B_n$. The sequence need not be $\mathcal{S}$-exact at C_n and if $H_nC \in \mathcal{S}$ for all n we say the sequence is $\mathcal{S}$-exact.

Definition 2.9. Let the sequence $I \to L \overset{\varphi}{\to} M \overset{\psi}{\to} N \to T$ be short $\mathcal{S}$-exact such that I,$T \in \mathcal{S}$, and $F : \mathcal{C} \to \mathcal{C}$ is covariant functor. We say F is left $\mathcal{S}$-exact if $I_1 \to FL \overset{F\varphi}{\to} FM \overset{F\psi}{\to} FN$ be $\mathcal{S}$-exact with $I_1 \in \mathcal{S}$. The morphism $F\psi$ is not, ingeneral a $\mathcal{S}$-epic. Dually, holds for right $\mathcal{S}$-exact functor. In general if the sequence $I_1 \to FL \overset{F\varphi}{\to} FM \overset{F\psi}{\to} FN \to T_1$ be $\mathcal{S}$-exact with $I_1, T_1 \in \mathcal{S}$ we say F is $\mathcal{S}$-exact functor.

Definition 2.10. A $\mathcal{S}$-injective resolution of an object A from an abelian category $\mathcal{C}$ is a $\mathcal{S}$-exact sequence $0 \to A \overset{\varepsilon}{\to} I^0 \to I^1 \to I^2 \ldots$ in $\mathcal{C}$, with

I^i,$i \geqslant 0$, $\mathcal{S}$-injective objects. Dually holds for $\mathcal{S}$-projective resolution if I_i,$i \geqslant 0$, will be $\mathcal{S}$-projective objects. Some times the above $\mathcal{S}$-injective resolution will be denoted by $A \xrightarrow{\varepsilon} I^\bullet$,$\varepsilon$ will be the augmentation morphism.

Remark 2.11. Let I^0 be a $\mathcal{S}$-injective object and $A \in \mathcal{S}$ so $A \xrightarrow{\varepsilon} I^0$ will be $\mathcal{S}$-monic. In general we can say a neccesary and sufficient condition for an object A from $\mathcal{C}$ to have a $\mathcal{S}$-injective resolution is that $A \in \mathcal{S}$, on the other hand A is subobject of a $\mathcal{S}$-injective object of $\mathcal{C}$. Similar results hold for $\mathcal{S}$-projective resolution (in that we must require that $A \in \mathcal{S}$, so A will be a quotient of a $\mathcal{S}$-projective object.

Definition 2.12. Let $F : \mathcal{C} \to \mathcal{C}$ be a left $\mathcal{S}$-exact functor. Assume that $\mathcal{C}$ has sufficiently many $\mathcal{S}$-injective objects. The right $\mathcal{S}$-derived functors $R^iF : \mathcal{C} \to \mathcal{C}, i \geqslant 0$ are defined as fallows; On objects; $R^iF(A) = H^i(F(I^\bullet))$ for $A \in Ob\mathcal{C}$ where $A \xrightarrow{\varepsilon} I^\bullet$ is a $\mathcal{S}$-injective resolution of A,$F(I^\bullet)$ is the complex obtained by termwise application on F to $I^\bullet$. On morphisms;$R^iF(\varphi) = H^i(F(\varphi^\bullet))$ for $\varphi : A \to A_1$, where $\varphi^\bullet : I^\bullet \to I_1^\bullet$ is an extension of φ to resolutions,$F(\varphi^\bullet) : F(I^\bullet) \to F(I_1^\bullet)$ is the corresponding morphism of resolutions.

Remark 2.13. Let F is left $\mathcal{S}$-exact functor. The calssical $\mathcal{S}$-derived functors R^iF (in our case, the right $\mathcal{S}$-derived functors), one of the most properties of $\mathcal{S}$-derived functors $R^iF : \mathcal{C} \to \mathcal{C}$ is that there exists a long $\mathcal{S}$-exact sequence

$$0 \to FA_1 \xrightarrow{F\varphi} FA \xrightarrow{F\psi} FA_2 \xrightarrow{\delta} R^1FA_1 \xrightarrow{R^1F\varphi} R^1FA \to \cdots \to R^nFA_1 \to R^nFA \to R^nFA_2 \to \ldots$$

for short $\mathcal{S}$-exact $0 \to A_1 \xrightarrow{\varphi} A \xrightarrow{\psi} A_2 \to 0$.

Definition 2.14. If $T = \mathrm{Hom}_{\mathcal{C}}(-, A)$ then we have $Ext^n_{\mathcal{C}}(C, A) = R^nT(C)$, and if $T = Hom_{\mathcal{C}}(C, -)$ then $Ext^n_{\mathcal{C}}(C, A) = R^nT(A)$.

Definition 2.15. The $\mathcal{S}$-injective dimension of M, denoted $\mathcal{S} - \mathrm{id}M$, is the minimal lengh of $\mathcal{S}$-injective resolution of M. If no $\mathcal{S}$-injective resolution of finite length exists, we say $\mathcal{S} - \mathrm{id}M = \infty$.

Theorem 2.16. $\mathcal{S} - \mathrm{id}M = 0$ *if and only if* M *is* $\mathcal{S}$*-injective object.*

Proof. Let M be $\mathcal{S}$-injective object and

$$0 \to M \to I^0 \to I^1 \to \ldots \quad (\dagger)$$

be a $\mathcal{S}$-injective resolution of M. Since M is $\mathcal{S}$-injective object we can say $I^0 = M$ then by $(\dagger)$ we have $I^1 = 0$. conversely, if $\mathcal{S} - \mathrm{id}M = 0$ then $0 \to M \to I^0 \to 0$ is $\mathcal{S}$-injective resolution of M and $M \cong I^0$. □

Remark 2.17. The dually of definition 2.15 and theorem 2.16 hold for $\mathcal{S}$-projective dimension of M, that we denoted it with $\mathcal{S} - \mathrm{pd}M$.

Theorem 2.18. *let* $\mathcal{C}$ *be an abelian category with null object* 0 *and* E *be* $\mathcal{S}$*-injective object. Then* $F : \mathcal{C} \to \mathcal{A}b$ *such that* $F(M) = \mathrm{Hom}_{\mathcal{C}}(M, E)$ *is* $\mathcal{S}$*-exact Functor.*

Proof. Let $0 \to M \xrightarrow{\varphi} N \xrightarrow{\psi} K \to 0$ be $\mathcal{S}$-exact sequence, we show the sequence

$$0 \to \mathrm{Hom}(K, E) \xrightarrow{\overline{\psi}} \mathrm{Hom}(N, E) \xrightarrow{\overline{\varphi}} \mathrm{Hom}(M, E) \to 0$$

is $\mathcal{S}$-exact. Let $f = \mathrm{Ker}\overline{\psi}$, then $\overline{\psi}f = 0$, and $f\psi = 0$ so $f = \mathrm{Coker}\psi \in \mathcal{S}$. If $f = \mathrm{Coker}\overline{\varphi}$ then $f\overline{\varphi} = 0$, and we have $\varphi f = \underline{0}$,$f \in \mathrm{Ker}\varphi$ so $f \in \mathcal{S}$. Now let $g \in Im\overline{\psi}$ and $f \in \mathrm{Hom}(K, E)$ such that $\overline{\psi}f = g$ so we will have $f\psi = 0$,$f\psi\varphi = g\varphi$,$g\varphi = 0$,$\overline{\varphi}g = 0$, so $g \in \mathrm{Ker}\overline{\varphi}$. Finally $\mathrm{Ker}\overline{\varphi}/Im\overline{\psi} \in \mathcal{S}$. □

Note that dually of Theorem 2.18 holds for $\mathcal{S}$-projective object P.

Theorem 2.19. *If M be $\mathcal{S}$-projective object then $\mathrm{Ext}^n_{\mathcal{C}}(M, N) \in \mathcal{S}$ for all $n \geq 1$ and every arbitrary object N.*

Proof. Let

$$0 \to P_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to M \to 0$$

be $\mathcal{S}$-projective resolution of M. Since M is $\mathcal{S}$-projective we can say $P_0 \cong M$ and $P_i = 0$,$i \geqslant 1$ and $\mathrm{Ext}^n_{\mathcal{C}}(M, N) \in \mathcal{S}$ for all $n \geqslant 1$. □

Theorem 2.20. *M is $\mathcal{S}$-projective object if and only if $\mathrm{Ext}^1_{\mathcal{C}}(M, N) \in \mathcal{S}$ for every arbitrary object N.*

Proof. If M be $\mathcal{S}$-projective object then we will have $\mathrm{Ext}^n_{\mathcal{C}}(M, N) \in \mathcal{S}$ for all $n \geqslant 1$. Conversely, if $\mathrm{Ext}^1_{\mathcal{C}}(M, N) \in \mathcal{S}$, we prove M is $\mathcal{S}$-projective object. We prove that $\mathrm{Hom}_{\mathcal{C}}(M, -)$ is $\mathcal{S}$-exact functor. Let

$$I \to N_1 \to N \to N_2 \to T$$

be $\mathcal{S}$-exact sequence, and the sequence

$$I_1 \to \mathrm{Hom}(M, N_1) \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N_2) \to \mathrm{Ext}^1(M, N_1) \to \dots$$

will be $\mathcal{S}$-exact. Now since I_1 and $\mathrm{Ext}^1(M, N_1)$ belong to $\mathcal{S}$, so $\mathrm{Hom}_{\mathcal{C}}(M, -)$ is $\mathcal{S}$-exact functor and M will be $\mathcal{S}$-projective object. □

Theorem 2.21. *Let M be an object in abelian category $\mathcal{C}$.$\mathcal{S} - \mathrm{pd}M < n$ if and only if $\mathrm{Ext}^n_{\mathcal{C}}(M, N) \in \mathcal{S}$.*

Proof. Let $\mathcal{S} - \mathrm{pd}M < n$, and

$$0 \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to M \to 0$$

be $\mathcal{S}$-projective resolution of M such that $P_n \cong 0$, then $\mathrm{Ext}^n_{\mathcal{C}}(M, N) = H^n(\mathrm{Hom}(P_n, M)) \in \mathcal{S}$. Conversely, we prove $\mathcal{S} - \mathrm{pd}M < n$, if $n = 0$ then $\mathrm{Ext}^0_{\mathcal{C}}(M, N) = \mathrm{Hom}(M, N)$ and if $N = M$ then $\mathrm{Hom}(M, M)) \in \mathcal{S}$, so $M \in \mathcal{S}$ and $\mathcal{S} - \mathrm{pd}M = -1$. If $n = 1$ and $\mathrm{Ext}^1_{\mathcal{C}}(M, N) \in \mathcal{S}$, then by theorem 2.19 $\mathcal{S} - \mathrm{pd}M = 0$. Let $n \geqslant 2$, and we prove $\mathcal{S} - \mathrm{pd}M < n$. If

$$0 \to X \to P_{n-2} \to P_{n-3} \to \cdots \to P_1 \to P_0 \to M \to 0$$

be $\mathcal{S}$-projective resolution of M. we must show X is $\mathcal{S}$-projective object, in fact $\mathrm{Ext}^1_{\mathcal{C}}(X,C) \in \mathcal{S}$ for arbitrary object C of $\mathcal{C}$. Define $A_s = Im(P_s \to P_{s-1})$ and we have the $\mathcal{S}$-exact sequences;

$$0 \to A_1 \to P_0 \to M \to 0$$

$$0 \to A_2 \to P_1 \to A_1 \to 0$$

finally

$$0 \to X \to P_{n-2} \to A_{n-2} \to 0.$$

By applying the functor $\mathrm{Ext}(-,C)$ and theorem 2.18 we will have isomorphisms;

$$\mathrm{Ext}^1(X,C) \cong \mathrm{Ext}^2(A_{n-2},C)$$

$$\mathrm{Ext}^2(A_{n-2},C) \cong \mathrm{Ext}^3(A_{n-3},C)$$

finally

$$\mathrm{Ext}^{n-1}(A_1,C) \cong \mathrm{Ext}^n(M,C)$$

since $\mathrm{Ext}^n(M,C) \in \mathcal{S}$ so $\mathrm{Ext}^1(X,C) \in \mathcal{S}$ then X is $\mathcal{S}$-projective object and $\mathcal{S} - \mathrm{pd}M = n-1 < n$. □

Theorem 2.22. *Let*

$$0 \to M_1 \to M \to M_2 \to 0$$

be $\mathcal{S}$-exact sequence.
(a) If M be $\mathcal{S}$-projective object then $\mathcal{S} - \mathrm{pd}M_2 = \mathcal{S} - \mathrm{pd}M_1 + 1$.
(b)$\mathcal{S} - \mathrm{pd}M_2 \leqslant 1 + Max\{\mathcal{S} - \mathrm{pd}M_1, \mathcal{S} - \mathrm{pd}M\}$.
(c) $\mathcal{S} - \mathrm{pd}M \leqslant Max\{\mathcal{S} - \mathrm{pd}M_1, \mathcal{S} - \mathrm{pd}M_2\}$.

Proof. (a) Let $\mathcal{S} - \mathrm{pd}M_2 = n$ then $\mathrm{Ext}^{n+1}_{\mathcal{C}}(M_2,N) \in \mathcal{S}$ and since M is $\mathcal{S}$-projective object then $\mathrm{Ext}^n_{\mathcal{C}}(M,N) \in \mathcal{S}$ for all $n \geqslant 1$, so by $\mathcal{S}$-exact sequence

$$\cdots \to \mathrm{Ext}^n(M,N) \to \mathrm{Ext}^n(M_1,N) \to \mathrm{Ext}^{n+1}(M_2,N) \to \ldots$$

we will have $\mathrm{Ext}^n_{\mathcal{C}}(M_1,N) \in \mathcal{S}$, and $\mathcal{S} - \mathrm{pd}M_1 \leqslant n-1 = \mathcal{S} - \mathrm{pd}M_2 - 1$ and

$$\mathcal{S} - \mathrm{pd}M_2 \geqslant \mathcal{S} - \mathrm{pd}M_1 + 1$$

now let $\mathcal{S} - \mathrm{pd}M_1 = n$ so $\mathrm{Ext}^{n+1}_{\mathcal{C}}(M_1,N) \in \mathcal{S}$ and by $\mathcal{S}$-exact sequence

$$\cdots \to \mathrm{Ext}^{n+1}(M_1,N) \to \mathrm{Ext}^{n+2}(M_2,N) \to \mathrm{Ext}^{n+2}(M,N) \to \ldots$$

we obtain $\mathrm{Ext}^{n+2}_{\mathcal{C}}(M_2,N) \in \mathcal{S}$ then $\mathcal{S} - \mathrm{pd}M_2 \leqslant \mathcal{S} - \mathrm{pd}M_1 + 1$.
(b)Let $\mathcal{S} - \mathrm{pd}M_1 = n$,$\mathcal{S} - \mathrm{pd}M = n$, then will be obtained $\mathrm{Ext}^k_{\mathcal{C}}(M_1,N) \in \mathcal{S}$,$\mathrm{Ext}^k_{\mathcal{C}}(M,N) \in \mathcal{S}$ for all $k \geqslant n+1$. Now the $\mathcal{S}$-exact sequence

$$\cdots \to \mathrm{Ext}^{n+1}(M_1,N) \to \mathrm{Ext}^{n+2}(M_2,N) \to \mathrm{Ext}^{n+2}(M,N) \to \ldots$$

will have that $\mathrm{Ext}^{n+2}_{\mathcal{C}}(M_2,N) \in \mathcal{S}$ so $\mathcal{S} - \mathrm{pd}M_2 \leqslant n+1$.
(c)Let $\mathcal{S} - \mathrm{pd}M_1 = n$,$\mathcal{S} - \mathrm{pd}M_2 = m$ with $n > m$, and apply the argument in proof of (b) repeatedly. □

Corollary 2.23. *Let*

$$0 \to M_1 \to M \to M_2 \to 0$$

be $\mathcal{S}$-exact sequence. Dually of theorem 2.22, holds for $\mathcal{S}$-injective dimension if $\mathcal{S}$-injective dimension of each M_1,M,M_2 is finite. In fact we have;

$$\mathcal{S} - \mathrm{id}M_1 \leqslant Max\{\mathcal{S} - \mathrm{id}M\mathcal{S} - \mathrm{id}M_2 + 1\},$$

$$\mathcal{S} - \mathrm{id}M \leqslant Max\{\mathcal{S} - \mathrm{id}M_2\mathcal{S} - \mathrm{id}M_1\},$$

$$\mathcal{S} - \mathrm{id}M_2 \leqslant Max\{\mathcal{S} - \mathrm{id}M\mathcal{S} - \mathrm{id}M_1 - 1\}.$$

Definition 2.24. Let M be an object in an abelian category $\mathcal{C}$. A $\mathcal{S}$-resolution of M is a $\mathcal{S}$-exact sequence

$$\cdots \to S_n \to S_{n-1} \to \cdots \to S_1 \to S_0 \to M \to 0$$

such that $S_i \in S$, for all $i \geqslant 0$.

Definition 2.25. An object M is said to have finite $\mathcal{S}$-dimension, and we write $\mathcal{S} - \dim M < \infty$ for short, if it has a $\mathcal{S}$-resolution of finite length. We set $\mathcal{S} - \dim 0 = -1$, and for $M \neq 0$ we define the $\mathcal{S} - \dim M$ of M as follow; For $n \geqslant 0$, we say that M has $\mathcal{S}$-dimension at most n, and write $\mathcal{S} - \dim M \leq n$,if and only if M has a $\mathcal{S}$-resolution of length n. If M has no $\mathcal{S}$-resolution of length, then we say that it has infinite $\mathcal{S}$-dimension, and we write $\mathcal{S} - \dim M = \infty$ so we have $M \in \mathcal{S}$ if and only if $\mathcal{S} - \dim M = 0$ or $M = 0$.

Theorem 2.26. *If M is $\mathcal{S}$-projective object then $M \in \mathcal{S}$.*

Proof. Since M is $\mathcal{S}$-projective object then $\mathcal{S}-\mathrm{pd}M = 0$ and $0 \to P_0 \to M \to 0$ will be $\mathcal{S}$-projective resolution of M, so $P_0 \cong M$, and $M \in \mathcal{S}$. □

Theorem 2.27. *Let M be object.$\mathcal{S} - \dim M < \mathcal{S} - \mathrm{pd}M$.*

Proof. If $M = 0$ equality holds and the inequality certainly holds if M has infinite $\mathcal{S}$-projective dimension. Assume that M is non-zero of finite $\mathcal{S}-\mathrm{pd}$,say n. We prove $\mathcal{S} - \dim M < n$, let $0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to M \to 0$ be $\mathcal{S}$-projective resolution of M such that all M_i,$i \geqslant 0$ are $\mathcal{S}$-projective, so by theorem 2.26 M_i,$i \geqslant 0$ belong to $\mathcal{S}$ and it will be $\mathcal{S}$-resolution of length n, so $\mathcal{S} - \dim M < n$. □

Theorem 2.28. *Let $0 \to L \to M \to N \to 0$ be a $\mathcal{S}$-exact sequence. If $\mathcal{S} - \dim N \leqslant n$, then $\mathcal{S} - \dim L \leqslant n$ if and only if $\mathcal{S} - \dim M \leqslant n$.*

Proof. First note that if $\mathcal{S}-\dim N \leqslant 0$ that is $N \in \mathcal{S}$, then $\mathcal{S}-\dim L \leqslant 0$ $\Leftrightarrow L \in \mathcal{S} \Leftrightarrow M \in \mathcal{S} \Leftrightarrow \mathcal{S}-\dim M \leqslant 0$. We now assume that $\mathcal{S}-\dim N \leqslant n$ and $n \geqslant 1$. Let

$$\cdots \to L_n \to L_{n-1} \to \cdots \to L_1 \to L_0 \to L \to 0$$

and

$$\cdots \to N_n \to N_{n-1} \to \cdots \to N_1 \to N_0 \to N \to 0$$

be two $\mathcal{S}$-resolutions of, respectively,L and N. Since $\mathcal{S}-\dim N \leqslant n$ we have $\mathcal{S}$-resolution

$$0 \to Y_n \to N_{n-1} \to \cdots \to N_1 \to N_0 \to N \to 0.$$

Note that

$$0 \to Z_n \to L_{n-1} \to \cdots \to L_1 \to L_0 \to L \to 0$$

will be $\mathcal{S}$-resolution of L if and only if

$$0 \to X_n \to L_{n-1} \oplus N_{n-1} \to \cdots \to L_1 \oplus N_1 \to L_0 \oplus N_0 \to M \to 0$$

be $\mathcal{S}$-resolution of M.

□

Definition 2.29. Let R is a commutative noetherian ring and $\mathfrak{p}$ be a prime ideal. As a general reference to homological and commutative algebra we use [3].$\mathcal{S}(R)$ and $\mathcal{S}(R_\mathfrak{p})$ be subcategory of R-modules and R-homomorphisms , $R_\mathfrak{p}$-modules and $R_\mathfrak{p}$-homomorphisms, respectively, such that belongs to serre class.

Lemma 2.30. *Let $M \in \mathcal{S}(R)$ be a finite R-module then $M_\mathfrak{p} \in \mathcal{S}(R_\mathfrak{p})$.*

Proof. Let $M \in \mathcal{S}(R)$ then $\mathcal{S}-\dim M = 0$, and $0 \to \mathcal{S}_0 \to M \to 0$ is $\mathcal{S}$-resolution of R-modules and R-homomorphisms, and we know that $S^{-1}: Mod(R) \to Mod(S^{-1}R)$ is $\mathcal{S}$-exact functor, so $0 \to (\mathcal{S}_0)_\mathfrak{p} \to M_p \to 0$ will be $\mathcal{S}$-resolution of $M_\mathfrak{p}$, and $\mathcal{S}-\dim_{R_\mathfrak{p}} M_\mathfrak{p} = 0$ then $M_\mathfrak{p} \in \mathcal{S}(R_\mathfrak{p})$ □

Theorem 2.31. *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be $\mathcal{S}$-exact sequence with finite R-modules in $\mathcal{S}(R)$, then $0 \to L_\mathfrak{p} \xrightarrow{f_\mathfrak{p}} M_\mathfrak{p} \xrightarrow{g_\mathfrak{p}} N_\mathfrak{p} \to 0$ will be $\mathcal{S}$-exact sequence in $\mathcal{S}(R_\mathfrak{p})$.*

Proof. Use lemma 2.29 and this fact that $S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$ for submodule N of M. □

Theorem 2.32. *let M be finite R-module . If $\mathcal{S}-\dim M \leqslant n$, then $\mathcal{S}-\dim_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant n$.*

Proof. If $n = 0$ then the result holds. Let $n \geqslant 1$, and

$$0 \to K_n \to S_{n-1} \to \cdots \to S_1 \to S_0 \to M \to 0$$

be $\mathcal{S}$-resolution of M such that $S_i \in \mathcal{S}(R)$ for all $i = 0, 1, 2, ..., n-1$. Now by applying $\mathcal{S}$-resolution

$$0 \to (K_n)_\mathfrak{p} \to (S_{n-1})_\mathfrak{p} \to \cdots \to (S_1)_\mathfrak{p} \to (S_0)_\mathfrak{p} \to M_\mathfrak{p} \to 0$$

we have $\mathcal{S} - \dim_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant n$. □

Acknowledgment. I would like to thank the referee for his/her useful comments on the paper.

References

[1] H. Zoschinger, *Koatomare Moduln*, Math. Z, **170**(1980), 221-232.

[2] H. Zoschinger, *Minimax Moduln*, J. Algebra, **102**(1986), 1-32.

[3] M. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics 60, Cambridge University Press (1998).

Received: May 20, 2013