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Torsion Purity in Ring and Modules

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Abstract

The aim of this paper is to relativize the concept of M- **purity** and σ-purity defined and studied by **Azumaya**[2] with respect to an arbitrary **hereditary torsion theory** given by a left exact torsion redical σ and also relate this concepts with the notions of σ- purity as given by **B. B. Bhattacharya** and **D. P. Choudhury**[3].

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1 Introduction

The notion of purity plays a fundamental role in the theory of abelian groups as well as in module categories. In the first section of this paper we examine the purities by torsion modules, finitely generated torsion modules and cyclic torsion modules. Work in this direction was initiated by **Walker**, **Stenstrom**, **Azumaya**[2], **B. B. Bhattacharya** and **D. P. Choudhury**[3]. In this there is an attempt to relativize the usual **Cohn** purity with respect to a torsion theory We also develope the theory of (M, σ)- purity and (μ, σ)- purity relative to a torsion theory with radical σ which is weaker than the usual purity and given a sufficient condition for these two coicide (when M is a left R- module

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and μ is an $i \times j$ matrix determined by M). In the second section of this present paper we relativize the concept of **weak** (M, σ)- purities corresponding to direct products of matrices of left modules which are row finite or those of right modules are column finite.

2 (M, σ)- purity

In this paper σ will denote a given left exact torsion radical and a torsion module means a module M for which $\sigma(M) = M$. Suppose that M, B and C are left R- modules.

Definition 2.1. An epimorphism $p : B \longrightarrow C$ is said to be $(\mathcal{M}, \sigma)$**- pure** if for each homomorphism $\varphi : M \longrightarrow C$ with image (φ) a torsion module that is $\varphi[M] \subseteq \sigma[B]$, there exists a homomorphism $\phi : M \longrightarrow B$ such that $po\phi = \varphi$

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow{\scriptstyle\varphi} & & \\ B & \xrightarrow{p} & C & \longrightarrow & 0 \end{array}$$

We may extend the lower sequence by taking the kernel of P denoted by A and refer the short exact sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{p} C \longrightarrow 0$$

as (M, σ)- pure.

If the epimorphism $p : B \longrightarrow C$ factorizes as

$$B \xrightarrow{g} N \xrightarrow{h} C$$

that is $p = (h0g)$ with g- epic, then we can easily see that whenever g and h are (M, σ)- pure then P is (M, σ)- pure.
Conversely in the above situation, if p is (M, σ)- pure then h is also (M, σ)- pure. If B is torsion then $p : B \longrightarrow C$ splits (that is kernel p is direct summand of A) if and only if p is (B, σ)- pure and this is equivalent to the condition that p is (M, σ)- pure for every left R- module M.

Given a row finite $I \times J$ matrix $\mu = (r_{ij})$, by a system of linear equations given by μ in a left module Y, we mean a system $\sum(r_{ij}x_j) = y_i$ where $y_i \in Y$ for each $i \in I$ and x_j $(j \in J)$ are unknowns.

Definition 2.2. We say that a submodule A is (μ, σ) -pure in a module B, if any system of linear equation $\sum_j r_{ij}x_j = a_i$ given by the row finite matrix μ in A, whenever solvable in B in the form $x_j = b_i$ for which there are left

ideals$D_i \in \mathcal{D}$ where $\mathcal{D}$ is the **Gabriel filter** of dense left ideals corresponding to the left exact torsion radical σ, such that $D_j b_j \subseteq A$, the system is also solvable in A that is there are $a'_j \in A$ with $\sum r_{ij} a'_j = a_i$ for every $i \in I$.
This exactly means that given vectors $(b_i) \in \prod_J B$ and $(a_i) \in \prod_I A$ and $\mu(b_j) = (a_i)$ with $D_j b_i \subseteq A$ for some $D_j \in \mathcal{D}$, there exists $(a'_j) \in \prod_I A$ such that $\mu(a'_j) = (a_i)$ where the vector $\mu(a'_j)$ is obtained by matrix product of the row finite matrix μ and column vector (a'_j). We may rephrase the above condition of A being (μ, σ) -pure in B or that B is a (μ, σ) -pure extension of A as follows.

Proof. If A is μ -pure in $\overline{A}$, and we have the relations $\sum r'_{ij} b_j = a_i$ with $b_j \in B, a_i \in A$ and $D_j b_j \subseteq A$ for some $D_j \in \mathcal{D}$, we have $D_j(b_j + A) = 0 \Rightarrow (b_j + A) \in \sigma(B/A) = \overline{A}/A$ so $b_j \in \overline{A}$. Thus by μ -purity of A in $\overline{A}$, there exists $a'_j \in A$ such that $\sum r_{ij} a'_j = a_i$ and hence A is (μ, σ) -pure in B. □

Corollary 2.3. *If a module M is given by a defining matrix μ, then a sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

of left R -modules is M -pure if and only if it is μ -pure.

Corollary 2.4. *If A is a submodule of a module B and $\overline{A}$ is the closure of A in B, then the followings are equivalent for a module M given by a row finite defining matrix μ:*

(i) A is (M, σ) -pure in B

(ii) A is μ -pure in $\overline{A}$

(iii) A is (M, σ) -pure in B

(iv) A is M -pure in $\overline{A}$

Proposition 2.5. *If $A \subseteq B \subseteq C$, then the following statements hold:*

(i) If A is M -pure in B and B is (M, σ) -pure in C then A is (M, σ) -pure in C.

(ii) If A is (M, σ) -pure in C then A is (M, σ) -pure in B.

(iii) If A is M -pure in C and B/A is (M, σ) -pure in C/A, then B is (M, σ) -pure in C.

(iv) If B is (M, σ) -pure in C then B/A is (M, σ) -pure in C/A.

Proof. (i) If $A \subseteq B \subseteq C$, then we have the following commutative diagram with exact rows and columns, the maps being given by inclusions or quotient maps

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{i_1} & B & \xrightarrow{p_1} & B/A & \longrightarrow & 0 \\
 & & {\scriptstyle I_A}\downarrow & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_4} & & \\
0 & \longrightarrow & A & \xrightarrow{i_3} & C & \xrightarrow{p_3} & C/A & \longrightarrow & 0 \\
 & & & & {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_4} & & \\
 & & & & C/B & \xrightarrow{I_{C/B}} & C/B & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By hypothesis, the upper squence is M -pure and the left vertical squence is (M, σ) -pure. To show that the second row is (M, σ) -pure. We take $f : M \longrightarrow C/A$ with $Im(f)$ torsion. Then $(p_1 o f) : M \longrightarrow C/B$ and its image is also a torsion module. As the left row is (M, σ) -pure, there exists $f' : M \longrightarrow C$ such that $(p_2 o f') = (p_4 o f)$. But by commutativity of the lower square $p_2 = (p_4 o p_3)$ therefore $p_4 o((p_3 o f') - f) = 0$. As $Im(i_4) = ker(p_4)$ therefore there exists $f" : M \longrightarrow B/A$ such that $((p_3 o f') - f) = (i_4 o f")$.
As the upper sequence is M -pure, $g : M \longrightarrow B$ will exist and satisfying $(p_1 o g) = f"$.
Now $p_3 o(f' - (i_2 o g)) = (p_3 o f') - (p_3 o i_2 o g) = (p_3 o f') - (i_4 o p_1 o g) = (p_3 o f' - i_4 o f") = f$.
Now $h : (f' o i_2 o g) : M \longrightarrow B$ and $p_3 o h = f$ showing that the second row is (M, σ) -pure that is A is (M, σ) -pure in C.

□

Proof. (ii) We consider the same diagram again and take any $f : M \longrightarrow B/A$

with $Im(f)$ torsion.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{i_1} & B & \xrightarrow{p_1} & B/A & \longrightarrow & 0 \\
 & & {\scriptstyle I_A}\downarrow & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_4} & & \\
0 & \longrightarrow & A & \xrightarrow{i_3} & C & \xrightarrow{p_3} & C/A & \longrightarrow & 0 \\
 & & & & {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_4} & & \\
 & & & & C/B & \xrightarrow{I_{C/B}} & C/B & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Then $(i_1of) : M \longrightarrow C/A$ and $Im(i_1of) \cong Im(f)$ is torsion. As the lower sequence is given to be (M, σ) -pure there exists $f' : M \longrightarrow C$ such that $(p_3of') = (i_4of)$. Then $(p_2of') = ()p_4op_3of' = (p4)oi_4of = 0$. Therefore f' factors through $ker(p_2) = B$ that is there exists $g : M \longrightarrow B$ such that $(i_2og) = f'$. Therefore $(i_4op_1og) = (p_3oi_2og) = (p_3of') = (i_4of)$. This gives $(p_1og) = f$ as i_4 is monic, This proves that A is (M, σ) -pure in B.

□

Proof. (iii) We again have the same set up but now to show that B is (M, σ) -pure in C, we take an arbitrary homomorphism $f : M \longrightarrow C/B$ with $Im(f)$ being torsion

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{i_1} & B & \xrightarrow{p_1} & B/A & \longrightarrow & 0 \\
 & & {\scriptstyle I_A}\downarrow & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_4} & & \\
0 & \longrightarrow & A & \xrightarrow{i_3} & C & \xrightarrow{p_3} & C/A & \longrightarrow & 0 \\
 & & & & {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_4} & & \\
 & & & & C/B & \xrightarrow{I_{C/B}} & C/B & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Now as the right column is (M, σ) -pure, we have $f' : M \longrightarrow C/A$ such that $(p_4 o f') = f$. As the lower row is M -pure, therefore there exists $g : M \longrightarrow C$ such that $(p_3 o g) = f'$. Therefore $f = (p_4 o f') = (p_4 o p_3 o g) = (p_2 o g)$ and hence the left sequence is (M, σ) =pure.

□

Proof. (iv)

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{i_1} & B & \xrightarrow{p_1} & B/A & \longrightarrow & 0 \\
 & & {\scriptstyle I_A}\downarrow & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_4} & & \\
0 & \longrightarrow & A & \xrightarrow{i_3} & C & \xrightarrow{p_3} & C/A & \longrightarrow & 0 \\
 & & & & {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_4} & & \\
 & & & & C/B & \xrightarrow{I_{C/B}} & C/B & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Given $f : M \longrightarrow C/B$ with $Im(f)$ a torsion module, there exists $f' : M \longrightarrow C$ such that $(p_2 o f') = f$. Then $h : M \longrightarrow C/A$ and $(p_4 o h) = (p_4 o p_3 o f') = (p_2 o f') = f$ and hence the right row is (M, σ) -pure.

□

3 Weak (M, σ)- purities

We now consider conditions weaker than (M, σ) and (μ, σ) -purities.

Definition 3.1. A sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

is said to be **weakly** (N, σ) **-pure** if the sequence

$$0 \longrightarrow N \otimes A \longrightarrow N \otimes \overline{A}$$

is exact, when N is a given right R -module. Here $\overline{A}$ is the closure of A in B, that is $\overline{A}/A = \sigma(B/A)$.

Definition 3.2. Given a column finite matrix $\nu = (s_{ij})$, an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

is said to be **weakly** (ν, σ) **-pure** if given a system of equations $\sum s_{ij}xja_i$ with $(x_j) \in \oplus_J B, (a_i) \in \oplus_I A$ such that for each $j \in J$, there is $D_j \in \mathcal{D}$ with $D_j x_j \subseteq A$, there exists $(a'_j) \in \oplus_J A$ with $\sum s_{ij}a'_j = a_i$. Note that we are restricting the vectors (x_j), (a_i) and (a'_j) to the corresponding direct sums of copies of B, A and A taken over J, I and J respectively, whereas in case of (μ, σ) -purity they could belong to the corresponding direct products. In case of I and J are finite then of course. the two ntions coincide.
Just as defining matrices of left modules are row finite, those of right modules are column finite. But in this case the defining matrix will be an $I \times J$ matrix if there is an exact sequence of right modules

$$\oplus_J R \xrightarrow{\nu} \oplus_I R \longrightarrow N \longrightarrow 0$$

where $\nu'(e_j) = \sum e_i s_{ij}$ and $\nu = (s_{ij})$ and to keep the sum finite, there should be at most finitely many non -zero (s'_{ij})'s for each j that is ν should be column finite.

Proposition 3.3. *For an exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

of left R -modules and a column finite matrix ν, the following statements are equivalent:

(i) The squence is weakly (ν, σ) -pure.

(ii) The squence is weakly (N, σ) -pure for a right R module N given by a column finite matrix ν.

(iii) A is weakly N -pure in $\overline{A}$ (where $\overline{A}/A = \sigma(B/A)$).

Proof. The notion of weak N -purity referred to in the statement (iii) above is the one defined in **Azumaya**[2] and $(ii) \iff (iii)$ follows from the definition of weak (N, σ) -purity.
By proposition 2, **Azumaya** [2], A is weakly N -pure in $\overline{A}$ if and only if A is weakly ν -pure in $\overline{A}$. Now the last condition means that given $(x_j) \in \oplus_J \overline{A}, (a_i) \in \oplus_I A$, with $\sum r_{ij}x_j = a_i$, there exists $a'_j \in \oplus_J A$ with $\sum r_{ij}a'_j = a_i$. But $x_j \in A$ means that $(x_j + A) \in \sigma(B/A)$ that is $D_j x_j \subseteq A$ and hence A is weakly ν -pure in $\overline{A}$ if and only if A is weakly (ν, σ) -pure in B. This proves the equivalence of (i) and (ii).

□

Theorem 3.4. *The following are equivalent for an exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

(i) The squence is (M, σ) -pure for all finitely presented modules M.

(ii) The squence is (μ, σ) -pure for all fnite matrices μ

(iii) A is pure in $\overline{A}$.

Proof. The proof this theorem is equivalent to the fact that the squence

$$0 \longrightarrow A \longrightarrow \overline{A} \longrightarrow \overline{A}/A \longrightarrow 0$$

remains after tensoring with every finitely presented module N. But this implies that it remains exact after tensoring with every module N as tensor products commute with direct limits and every modules a direct limit of finitely presented modules. □

References

[1] F.W. Anderson and K. R. Fuller, *Rings and categories of modules*, Springer Verlag, New York, (1974).

[2] Garo Azumaya, *Finite Splitness and finite projectivity*, Journal of Algebra**106**, 114–134 (1972).

[3] B. B. Bhattacharya and D. P. Choudhury, *Purities relative to a torsion theory*, Indian J. Pure appl. Math. **14**(4), 554–564(1983).

[4] D.P. Choudhury and K. Tewari, *Tensor purities, cyclic quasi-projectives and cocyclic copurity*, Commn. in Algebra **7**, 1559–1572 (1979).

[5] S. E. Dickson,*A torsion theory for Abelian catogories*, Trans. Amer. Maths. Soc. **121**, 223–235(1966).

[6] A. Hattori,*A torsion theory for modules over general rings*, Nagoya Maths.J. **17**, 147–158(1960).

[7] J. Jans,*Some aspects of torsion*, Pacific J. Maths. **15**, 1249–1259(1965).

[8] J. Lambek, *Torsion theories, additive semantics and rings of quotients*, Lecture notes in Math. Vol.**177**, Springer- Verlag, New York, 1971.

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