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Some Examples and Remarks Concerning Groups

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Abstract

We present some examples and remarks which may be helpful to those who are dealing with an abstract algebra or a first semester group theory course. Alternating groups, dihedral groups, and symmetric groups of small orders are treasure troves of elementary examples and counter examples concerning groups.

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1. Introduction

The goal of this paper is to point out some examples and remarks which will be helpful to those who are dealing with an abstract algebra or a first semester group theory course. In abstract and theoretical mathematics such as abstract algebra and group theory, examples are not only very helpful, but we believe are essential in dissecting a concept or analyzing a theorem. However, we need to keep in mind that while we can use examples to disprove a statement, or we may even use examples to conjecture, we cannot prove a statement by examples. As the saying goes: even a broken clock is correct twice a day.

Throughout the paper our definitions, notations, terminologies, and symbols will be standard. We use S_n and A_n to represent symmetric and alternating groups of degree n. The dihedral group of order n (it has $2n$ elements) is represented by D_n, and the infinite dihedral group is represented by D_∞. We keep in mind that $D_3 = S_3$ (the smallest non-abelian group) and $D_2 = V_4$, where V_4 is the Klein's group. $|G|$ represents the order of a group G and $|a|$ represents the order of an element $a \in G$. We caution the reader that we use 1 for both the identity element and the identity subgroup. In this article we will be dealing with discrete functions, and we note that we use definitions as "if and only if" statements.

2. Examples and Remarks

1. In any group, the empty set generates the identity subgroup, for the empty set is contained in every subgroup and the intersection of all subgroups is 1.

2. Not all subsets of a group are subgroups. For example, $\{a, b\}$ is a subset of $V_4 = \{1, a, b, ab\}$, but it is not a subgroup.

3. No group is the union of two of its proper subgroups.

4. The union of two subgroups may not be a subgroup. However, the union of three subgroups may be a subgroup. An example for both cases is V_4.

5. A regular polygon with n sides could generate both D_n and C_n (the cyclic group of order n).

6. A group with no nontrivial proper subgroup is a cyclic group of prime order.

7. Any quotient of any cyclic group is finite.

8. Every nontrivial element of a group could generate the group. An example is any cyclic group of prime order.

9. All proper subgroups of a non-cyclic group could be cyclic. An example is V_4.

10. All proper subgroups of a non-abelian group could be abelian. An example is S_3.

11. A group may have more subgroups than elements. An example is V_4.

12. A group may have more proper subgroups than elements. An example is D_4.

13. A cyclic group may be generated by more than a single element. If G is a group generated by a, then $G =< a^m, a^n >$, provided m and n are relatively prime, because there are integers r and s such that $rm + sn = 1$.

14. In a cyclic group the equation $x^n = 1$ has at most n solutions, while in a non-cyclic group it may have more than n solutions. For example, in V_4 the equation $x^2 = 1$ has 4 solutions.

15. In an abelian group all elements that satisfy the equation $a^n = 1$ ($n \in \mathbb{N}$) form a subgroup. This may not be true for non-abelian groups. For example, all elements of the form $a^2 = 1$ in D_4 do not form a group.

16. If G is an abelian group, then $(ab)^{-1} = a^{-1}b^{-1}$, $\forall a, b \in G$. Also, this may be true for some non-abelian groups as well. An example is D_4.

17. If $H < G$, and $a \in G \backslash H$, then aH may not necessarily be a subgroup. Also, if $H, K \leq G$, and $H \nleq K$, then this does not necessarily mean that $K < H$.

18. If $a, b \in G$ are two nontrivial elements, then $|a| = |a^{-1}|$ and $|ab| = |ba|$.

19. If $a, b \in G$ are two nontrivial elements, then $|ab| \neq |a||b|$. For example, if $a, b \in V_4$, then $|a||b| = 4 \neq |ab| = 2$. Moreover, if $a, b \in D_\infty$, then $|a| = |b| = 2$, while $|ab| = \infty$.

20. The infinite dihedral group D_∞ is generated by 2 elements of order 2 that do not commute, while V_4 is generated by two elements of order 2 that do commute. Behold the power of commutativity!

21. The infinite dihedral group D_∞ is the free product of 2 cyclic groups of order 2, while V_4 is the direct product of 2 cyclic groups of order 2.

22. If $H \leq G$, then $H^n = H$ for all positive integers and $(aH)^{-1} = Ha^{-1}$, for all $a \in G$.

23. If G/N is abelian, then G itself may not abelian. For example, S_3/A_3 is abelian, but S_3 is not.

24. If $|G : H| = 2$, then $H \lhd G$. However, the converse may not be true, for all subgroups of abelian groups are normal and they may not necessarily be of index 2. For example, $H =< a^3 >$ is of index 3 in $G =< a : a^9 = 1 >$.

25. In any group the equation $x^n = 1$ ($n \in \mathbb{N}$) has at least one solution. In some groups it may have (*i*) exactly n solutions, (*ii*) less than n solutions, (*iii*) more than n solutions, or (*iv*) infinitely many solutions.

Note 2.1. We recall that if H is a subgroup of a finite group G, then $|H|||G|$ (Lagrange's theorem). Also, we recall that if $|G| < \infty$, and p is a prime number such that $p||G|$, then G has an element of order p, and hence a subgroup of order p (Cauchy's theorem).

26. In general the converse of Lagrange's theorem may not be true. For example, A_4 does not have a subgroup of order 6. However, the converse is true if the conditions of Cauchy's theorem are satisfied. In fact the converse of Lagrange's theorem is true for a larger class of groups, namely finite cyclic groups. Also, note that the converse of Lagrange's theorem is true for the non-cyclic group V_4, for it has subgroups of orders $1, 2$, and 4. Moreover, if p is a prime number and k is a positive integer such that $p^k||G|$, then G has a subgroup of order p^k.

27. Normality of subgroups is not transitive. That is, if $H \lhd K \lhd G$, then this does not necessarily mean that $H \lhd G$. An example is D_4.

28. If $H, K \leq G$ such that $H \simeq K$, then this may not imply that $G/H \simeq G/K$. Also, we may have $G/H \simeq G/K$ without H and K being isomorphic. An example for both cases is D_4.

29. If $|G| < \infty$ such that $H \leq G$, and $a \in G$, then $|aH|||a|$ and $|aH|=|H|$.

30. If $a, b \in G$ and $H \leq G$, then we may have $aH = bH$, but $|a| \neq |b|$. An example is Z_6 (the group of integers modulo 6).

31. If $H \leq G$, and $a, b \in G$, and $aH = bH$, then this does not necessarily mean that $a = b$.

32. If $H \leq G$, and $a, b \in G$, and $aH = bH$, then this does not necessarily mean that $a^2H = b^2H$. An example is D_4.

33. If $H, K \leq G$ such that H and K are conjugate, then$H \simeq K$. The converse may not be true. An example is V_4.

34. If a, b are nontrivial and if $ab \in H \leq G$, then it does not necessarily mean that $a \in H$. For example, if $G =< a >$ and $H =< a^2 >$, then $aa \in H$, but $a \notin H$.

35. If x and y are conjugate elements in G, then $|x| = |y|$. To show that the converse may not be true, consider an abelian group of order 3 or higher.

36. It is possible that both H and G/H are abelian, without G itself being abelian.

37. If $H, K \leq G$ and $HK = KH$ then, it does not necessarily mean that $hk = kh$ for all $h \in H$ and all $k \in K$. Also, if $h' \in H$ and $k' \in K$ and $hk = h'k'$, then again this does not necessarily mean that $h = h'$ and $k = k'$.

38. A group may be isomorphic to its group of automorphisms. An example is S_3.

39. If G is a finitely generated abelian group, then all of its subgroups are finitely generated and abelian. However, if G is finitely generated, but not abelian, then it may have subgroups which are not finitely generated. An example is $G =< a, b : a^{-1}ba = b^2 >$.

40. We note that the symmetric group of the outer shell of the crosssection of the AIDS virus is D_{28}, and soccer enthusiasts should know that the rotation group of a soccer ball is A_5.

3. Closing Remarks

1. It is clear that $\xi = \{(1,1),(-1,-1)\}$ is a function. It is interesting to observe that $\xi = \xi^{-1} = \frac{1}{\xi}$. Are there other functions with this property? In the context of this article, what other properties does this function have?

2. In Example 25 which groups (or classes of groups) satisfy (i), (ii), (iii), and/or (iv)?

3. We know that all subgroups of abelian groups are normal. Can you think of examples of non-abelian groups where all subgroups are normal?

4. If G is an abelian group with $2n$ elements, where n is odd, then G has exactly one subgroup of order 2. One could apply Sylow theorems to get an immediate result. How would you answer this question without using Sylow theorems?

5. If P is the free product with amalgamations of any collection $\{C_\gamma\}_{\gamma\in\Gamma}$ of infinite cyclic groups, where Γ is an indexing set of cardinality greater than one, then we can construct an element $h \in P$ and a subgroup $T < P$ such that $|P : T| = \infty$, but $|P :< h, T > | < \infty$. Is h unique? Is T is unique?

6. We provided an example or a counter example for most of the statements that we made in Section 2. Can you think of other examples or classes of examples or counter examples? Can you also provide examples or some justification for those statements for which we did not provide an example or a counter example?

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