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Multiplication Modules on Arithmetical Rings

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Abstract

In this paper we investigate some results of multiplication modules on arithmetical rings. We introduce the concept of quasi-copure submodule of an R-module M and we will give some properties of them.

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1 Introduction

Throughout this work R is a commutative ring with identity and M is a unitary R-module. A submodule N of an R-module M is a distributive submodule if $N \cap (X + Y) = (N \cap X) + (N \cap Y)$, for all submodules X, Y of M. An R-module M is said to be distributive if every submodule of M is a distributive submodule. A commutative ring R is said to be arithmetical if any of the following equivalent conditions holds:

1.The localization of R at $\mathfrak{m}$ is a valuation ring for every maximal ideal $\mathfrak{m}$.

2.For all ideals , I, J, K of R we have $I \cap (J + K) = (I \cap J) + (I \cap K)$.

3.For all ideals I, J, K of R; $I + (J \cap K) = (I + J) \cap (I + K)$,

The last two conditions both say that the lattice of all ideals of R is distributive. An arithmetical domain is called a Prüfer domain. We know that every Prüfer domain and every Dedekind domain is an arithmetical ring. A

multiplication module M on an arithmetical ring R is distributive [[8], Theorem 1.2]. A finitely generated faithful R-module M is distributive if and only if M is a multiplication module and R is arithmetical [[8], Theorem 1.3].
The commutative ring R with identity is arithmetical ring if and only if, for any maximal ideal $\mathfrak{m}$, the ideals of the local ring $R_{\mathfrak{m}}$ are totally ordered by set inclusion, see [[3], Lemma 1].

2 Pure and copure submodules

Definition 2.1. *An R-module M is called multiplication if for every submodule N of M, there exists an ideal I of R such that $N = IM$. We can take $I = [N :_R M] = [IM :_R M]$ and I is called the presentation ideal of N, see [1].*

Definition 2.2. *A submodule N of M is said to be pure if $IN = N \cap IM$ for every ideal I of R, and we denote this concept by $N <_p M$. Moreover, N is said to be copure if $[N :_M I] = N + [0 :_M I]$ for every ideal I of R. An R-module M is said to be fully pure (resp. fully copure) if every submodule of M is pure (resp. copure), see [4], [6], [7].*

Definition 2.3. *Let N and K be two submodules of M. The product of N and K is defined by $NK = [N :_R M][K :_R M]M$. Also, the coproduct of N and K is defined by $C(NK) = [0 :_M Ann_R(N)Ann_R(K)]$.*

Definition 2.4. *A submodule N of M is said to be idempotent (resp. coidempotent) if $N = N^2$ (resp. $N = C(N^2) = [0 :_M Ann^2_R(N)]$). Moreover, M is said to be fully idempotent (resp. fully coidempotent) if every submodule of M is idempotent (resp. coidempotent).*

Lemma 2.5. *Let R be an arithmetical ring and M a multiplication R-module and N_1, N_2, K are submodules of M, then*
i) $(N_1 \cap N_2) + K = (N_1 + K) \cap (N_2 + K)$.
ii) $K \cap (N_1 + N_2) = (K \cap N_1) + (K \cap N_2)$.
Proof: *i) Let $N_1 = IM$, $N_2 = JM$, $K = TM$ for some ideals I, J, T of R, then* $(N_1+K)\cap(N_2+K) = (IM+TM)\cap(JM+TM) = ((I+T)M)\cap((J+T)M) = [(I+T)\cap(J+T)]M = [(I\cap J)+T]M = (I\cap J)M+TM = (IM\cap JM)+TM = (N_1 \cap N_2) + K$.
ii) It is similar to part (i). Therefore if M be a multiplication module on a arithmetical ring, then M is distributive module.

Corollary 2.6. *Let R be an arithmetical ring and M be a multiplication R-module. Let $\{N_\lambda\}_{\lambda\in\Lambda}$ and K are submodules of M, then*

$$(\bigcap_{\lambda\in\Lambda} N_\lambda) + K = \bigcap_{\lambda\in\Lambda}(N_\lambda + K)$$

3 Preliminary Notes

Definition 3.1. *Let N be a submodule of M, then M-radical of N is denoted by $rad_M(N)$ and it is the intersection of all prime submodules containing N. Therefore $rad_M(N) = \bigcap_{N \leq P \in Spec(M)} P$. The submodule N is called a radical submodule of M if $rad_M(N) = N$.*

Theorem 3.2. *Let M is a multiplication module and N is a copure submodule of M, then N is idempotent.*
Proof: *Since M is a multiplication module, hence $N = [N :_R M]M$ and also N is copure, then for every ideal I of R, $[N :_M I] = N + [0 :_M I]$. We take $I = [N :_R M]$, then $M = [N :_M [N :_R M]] = N + [0 :_M [N :_R M]]$ therefore*

$$N = [N :_R M]M = [N :_R M]\left\{N + [0 :_M [N :_R M]]\right\} = [N :_R M]N$$
$$= [N :_R M][N :_R M]M = [N :_R M]^2 M = N^2$$

Definition 3.3. *A submodule N of M is quasi-copure submodule if every proper prime submodule P contain N be a copure submodule of M.*

It is clear that if M be a multiplication R-module and N and K be quasi-copure submodules of M, then $N \cap K$ and $N + K$ are also quasi-copure submodules of M.

Example 3.4. *We consider $M = Z_8 \oplus Z_6$ as a Z-module and $N = \langle \bar{2} \rangle \oplus \langle \bar{3} \rangle$ as a submodule of M. One can check that N is not copure submodule of M and also $L = Z_8 \oplus \langle \bar{3} \rangle$ and $K = \langle \bar{2} \rangle \oplus Z_6$ are proper prime submodules of M contained N where both of them are copure submodules of M, therefore N is a quasi-copure submodule of M.*

Theorem 3.5. *Let R be an arithmetical ring and M a multiplication R-module and N is a quasi-copure submodule of M, then $rad_M(N)$ is a copure idempotent submodule of M. In particular, if N be a radical submodule of M, then N is a copure idempotent submodule of M.*
Proof: *By Corollary 2.6 we have*

$$\begin{aligned}[rad_M(N) :_M I] &= [\bigcap_{N \leq P \in Spec(M)} P :_M I] = \bigcap_{N \leq P \in Spec(M)} [P :_M I] \\ &= \bigcap_{N \leq P \in Spec(M)} (P + [0 :_M I]) = (\bigcap_{N \leq P \in Spec(M)} P) + [0 :_M I] \\ &= rad_M(N) + [0 :_M I]\end{aligned}$$

Now since M is a multiplication module and $rad_M(N)$ is a copure submodule of M, then by Theorem 3.2, $rad_M(N)$ is idempotent.
Now let N be a radical submodule of M, i.e. $N = rad_M(N)$, hence by first part N is a copure submodule of M and by Theorem 3.2, N is idempotent.

Now by ([5], Theorem [15]), we have the following results:

Corollary 3.6. *Let R be an arithmetical ring and M a multiplication R-module and N is a submodule of M.*
i) If $[N : M]$ be a radical ideal, then N is copure idempotent submodule of M.
ii) If M be f.g. where $I + ann(M)$ is a radical ideal of R, then $N = IM$ is a radical submodule of M and hence copure idempotent submodule of M.
iii) If N and K are quasi-copure submodules of M, then $rad_M(N \cap K)$ is a copure idempotent submodule of M.

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