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# The Automorphism Group of the Group of Unitriangular Matrices over a Field[1]

**Ayan Mahalanobis**

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road, Pashan, Pune 411008, India
ayan.mahalanobis@gmail.com



**Abstract.** This paper finds a set of generators for the automorphism group of the group of unitriangular matrices over a field. Most of this paper is an exposition of the work of V.M. Levčhuk, part of which is in Russian. Some proofs are of my own.

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## 1. Introduction

The automorphism group of the group of unitriangular matrices over a field was studied by many [2–4]. In this direction, the first paper was in Russian, published by Pavlov in 1953. Pavlov studies the automorphism group of unitriangular matrices over a finite field of odd prime order. Weir [4] describes the automorphism group of the group of unitriangular matrices over a finite field of odd characteristic. Maginnis [3] describes it for the field of two elements and finally Levčhuk [2] describes the automorphism group of the group of unitriangular matrices over an arbitrary ring.

In this expository article, we shall study the automorphism group of the group of unitriangular matrices over an arbitrary field **F**. There are two most commonly used non-abelian finite $p$-groups in the literature. One is the group

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of unitriangular matrices over a field and the other is the extra-special $p$-groups. Weir [4] and Levčhuk [2] worked on the automorphism group of the unitriangular matrices. Weir's work is unintelligible and Levčhuk works with unitriangular matrices over an arbitrary ring. The generality makes his work hard to follow. There is no easy-to-read exposition that presents the automorphism group of the unitriangular matrices over a field. We hope that this paper will fill that gap. The structure of the paper is the following: we find the maximal abelian normal subgroup of the group of unitriangular matrices; any automorphism will permute the maximal abelian normal subgroups. This gives us the central and working idea about the automorphisms and leads us to a proof of the last theorem (Theorem 3.3). There is a very interesting display of an interplay between an algebra, its circle group and its Lie ring in this paper.

To start, we denote the algebra of all lower niltraingular matrices over $\mathbf{F}$, of size $d$, by $\mathrm{NT}(d, \mathbf{F})$. This is the set of all matrices that have zero on and above the diagonal, and arbitrary field element (possible non-zero) below the main diagonal. It is known to be a *nilpotent algebra*, $M^d = 0$, for all $M \in \mathrm{NT}(d, \mathbf{F})$. Where 0 is the zero matrix of size $d$. The general method, that we describe below, works only when $d$ is greater than 4. The case of $d = 3$ and $d = 4$ can be computed by hand and was done by Levčhuk [2]. Henceforth, we assume that $d \geq 5$.

One can define two operations on the set $\mathrm{NT}(d, \mathbf{F})$.

**:** The first operation is $\circ$, defined as $a \circ b = a + b + ab$.

**:** The second operation is $*$, defined as $a * b = ab - ba$.

It is known that $(\mathrm{NT}(d, \mathbf{F}), \circ)$ is isomorphic to $\mathrm{UT}(d, \mathbf{F})$, the group of (lower) *unitriangular matrices*. The isomorphism being $x \mapsto 1 + x$, where 1 is the identity matrix of size $d$. This group is also known as the *associated group* of the ring $\mathrm{NT}(d, \mathbf{F})$. In this paper, we will denote the associated group of $\mathrm{NT}(d, \mathbf{F})$ by $\mathrm{UT}(d, \mathbf{F})$.

The second operation is a Lie bracket, it is known that $(\mathrm{NT}(d, \mathbf{F}), +, *)$ is a Lie algebra. In this paper, we will denote this Lie algebra by $\mathrm{NT}^*(d, \mathbf{F})$. It is not hard to see, in the light of Equations 3 and 4 later, that this Lie algebra is the same as the graded Lie Algebra of the group $\mathrm{UT}(d, \mathbf{F})$.

For $i > j$ and $x \in \mathbf{F}$, we define the matrix unit $xe_{i,j}$ to be the $d \times d$ matrix with $x$ in the $(i, j)$ position and 0 everywhere else.

The *de ning relations* in these three algebraic objects are the relations in the field $\mathbf{F}$ and the following:

The algebra $\mathrm{NT}(d, \mathbf{F})$.

$$(xe_{i,j})(ye_{k,l}) = \begin{cases} xye_{i,l} & \text{whenever} \quad j = k \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The group $\mathrm{UT}(d, \mathbf{F})$.

$$(xe_{i,j}) \circ (ye_{i,j}) = (x + y)e_{i,j} \tag{2}$$

$$[xe_{i,j}, ye_{k,l}] = \begin{cases} xye_{i,l} & \text{whenever} \quad j = k \\ -xye_{k,j} & \text{whenever} \quad i = l \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The Lie algebra $\mathrm{NT}^*(d, \mathbf{F})$.

$$(xe_{i,j}) * (ye_{k,l}) = \begin{cases} xye_{i,l} & \text{whenever} \quad j = k \\ -xye_{k,j} & \text{whenever} \quad i = l \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

From the Relation 3, it follows, that a set of generators for $\mathrm{UT}(d, \mathbf{F})$, is of the form $xe_{i+1,i}$, $x \in \mathbf{F}$ and $i = 1, 2, \ldots, d-1$. This is actually a set of *minimal* generators. Since the commutator relation and the relation in the Lie algebra are the same, the same set acts as generators for the Lie algebra as well.

Define

$$\Gamma_k = \left\{ M = \sum m_{i,j} e_{i,j} \in \mathrm{UT}(d, \mathbf{F}); \;\; m_{i,j} = 0, \;\; i - j < k \right\},$$

in other words, the $\Gamma_1 = \mathrm{UT}(d, \mathbf{F})$. The subgroup $\Gamma_2$ is the commutator of $\mathrm{UT}(d, \mathbf{F})$. It consists of all lower niltriangular matrices with the first subdiagonal entries zero. The first subdiagonal can be specified by all entries $(i, j)$ with $i - j = 1$. Similarly $\Gamma_2$ consists of all matrices with the first two subdiagonals zero and so on. It follows that $\Gamma_d = 0$.

It follows from Relation 3, if $i - j = k_1$ and $k - l = k_2$ and $[xe_{i,j}, ye_{k,l}]$ is non-zero, then the commutator is $xye_{i,l}$ or $xye_{k,j}$. In both these cases, $i - l$ or $k - j$ equals $k_1 + k_2$. Taking these into account, one can prove the next proposition.

**Proposition 1.1.** In $\mathrm{UT}(d, \mathbf{F})$, the lower central series and the upper central series are identical and is of the form $\mathrm{UT}(d, \mathbf{F}) = \Gamma_1 > \Gamma_2 > \ldots > \Gamma_{d-1} > \Gamma_d = 0$.

There is an interesting and useful connection between the normal subgroups of $\mathrm{UT}(d, \mathbf{F})$ and ideals of $\mathrm{NT}^*(d, \mathbf{F})$. The connection can be motivated by a simple observation: let $1 + L$ be in $\mathrm{UT}(d, \mathbf{F})$, i.e., $L \in \mathrm{NT}(d, \mathbf{F})$. Then

$$(1 + e_{ij})^{-1} (1 + L) (1 + e_{ij}) = 1 + L + (L * e_{ij}), \tag{5}$$

which implies that under suitable conditions, elements in a normal subgroup of $\mathrm{UT}(d, \mathbf{F})$ are closed under Lie bracket. Conversely, under suitable condition, an ideal of $\mathrm{NT}^*(d, \mathbf{F})$ is a normal subgroup of $\mathrm{UT}(d, \mathbf{F})$.

Furthermore one should also notice, if a subgroup $H$ of $\mathrm{UT}(d, \mathbf{F})$ is abelian, then we have $(1 + L)(1 + M) = (1 + M)(1 + L)$ for $1 + L, 1 + M \in H$; which implies that $L * M = 0$, i.e., if a subgroup is abelian and an ideal, then that ideal is abelian as well and vice versa.

Notice that, in the motivation above, we have represented an element of the group $\mathrm{UT}(d, \mathbf{F})$ as $1 + L$, where $L \in \mathrm{NT}(d, \mathbf{F})$. This is not necessary, we can use $L$ and the operation $\circ$. However, since $1 + L$ makes the group operation matrix multiplication, this makes our motivation transparent. From now on, elements in $\mathrm{UT}(d,\mathbf{F})$ will be represented as elements of $\mathrm{NT}(d,\mathbf{F})$, with the operation $\circ$.

For $i>j$, let us define $\mathbf{N}_{i,j}$ to be the subset of $\mathrm{NT}(d,\mathbf{F})$ with all rows less then the $i^{\text{th}}$ row zero and all columns greater than the $j^{\text{th}}$ column zero. It is a rectangle and Weir [4] calls it a *partition* subgroup. It is straightforward to see that $\mathbf{N}_{i,j}$ is an abelian (two-sided) ideal of the ring $\mathrm{NT}(d,\mathbf{F})$. From this it follows that $\mathbf{N}_{i,j}$ is an abelian ideal of $\mathrm{NT}^*(d,\mathbf{F})$.

**Lemma 1.2.** If $H$ is a maximal abelian normal subgroup of $\mathrm{UT}(d,\mathbf{F})$ or a maximal abelian ideal of $\mathrm{NT}^*(d,\mathbf{F})$, then

- $H^2 \subseteq H$.
- $H^2 \subseteq \mathbf{N}_{d,1}$.
- $\alpha\gamma\beta+\beta\gamma\alpha=0$ for $\alpha,\beta\in H$ and $\gamma\in\mathrm{NT}(d,\mathbf{F})$.

*Proof.* From maximality, it follows that $H$ contains the annihilator of the ring $\mathrm{NT}(d,\mathbf{F})$, i.e., the subset $\{x \mid xy=0=yx, \text{ for all } y\in\mathrm{NT}(d,\mathbf{F})\}$. We now show that $H^2$ is contained in the annihilator. We only work with the associated group, the proof for Lie algebra is identical.

Since $H$ is normal, for any $\alpha,\beta\in H$ and $\gamma=xe_{ij}\in\mathrm{NT}(d,\mathbf{F})$, $\alpha$ commutes with $(-xe_{ij})\circ\beta\circ(xe_{ij})$. This implies that

$$\alpha\circ(\beta+\beta(xe_{ij})-(xe_{ij})\beta)=(\beta+\beta(xe_{ij})-(xe_{ij})\beta)\circ\alpha$$

Since $H$ is abelian, $\alpha\beta=\beta\alpha$,

$$(\alpha\beta(xe_{ij})+(xe_{ij})\beta\alpha)-(\alpha(xe_{ij})\beta+\beta(xe_{ij})\alpha)=0$$

Now notice that the matrix represented in the first parenthesis has non-zero elements only on the $i^{\text{th}}$ row and the $j^{\text{th}}$ column. On the other hand the matrix represented by the second parenthesis has both the $i^{\text{th}}$ row and the $j^{\text{th}}$ column zero. Hence the equality is possible only when both the matrices are zero.

This implies that $\alpha\beta(xe_{ij})=0=(xe_{ij})\beta\alpha$ for $i>j$ and $i=2,3,\ldots,d$ and $j=1,2,\ldots,d-1$. It is also clear that $\alpha\gamma\beta+\beta\gamma\alpha=0$ for $\gamma=xe_{i,j}$.

Notice that for any $d\times d$ matrix $A$, $Ae_{i,j}$ is the matrix with only the $j^{\text{th}}$ column non-zero and the contents are the contents of the $i^{\text{th}}$ column. Thus $\alpha\beta(xe_{ij})=0$ implies that the $i^{\text{th}}$ column of $\alpha\beta$ is zero for $i=2,3,\ldots,d$. Similarly one can show, that the $j^{\text{th}}$ row is zero for $j=1,2,\ldots,d-1$. Then it follows that $H^2\subseteq\mathbf{N}_{d,1}$.

Since any matrix in $\mathrm{NT}(d,\mathbf{F})$ can be written as a linear combination of elementary matrices $xe_{i,j}$, the proof that $H^2$ is contained in the annihilator is complete. Furthermore, since $H$ is closed under addition, we have $\alpha\gamma\beta+\beta\gamma\alpha=0$, for all $\gamma\in\mathrm{NT}(d,\mathbf{F})$. •

**Theorem 1.3** (Levčhuk, 1976)**.** A maximal abelian normal subgroups of $\mathrm{UT}(d,\mathbf{F})$ is also a maximal abelian ideals of $\mathrm{NT}^*(d,\mathbf{F})$ and vice versa.

*Proof.* Let $N$ be a maximal abelian normal subgroup of $\mathrm{UT}(d,\mathbf{F})$. Then construct the subgroup

$$N' = \langle N \cup \{\alpha - \beta\}, \quad \alpha, \beta \in N \rangle .$$

Clearly $N'$ is an abelian subgroup. Since matrix multiplication distributes over addition, the subgroup $N'$ is normal. Since $N$ is contained in $N'$, the maximality implies that $N = N'$ and so $N$ contains the sum of any two of its elements.

To show that it is closed under Lie bracket, notice that, $N$ is normal implies $(-xe_{ij}) \circ \alpha \circ (xe_{ij})$ is in $N$. This implies that $\alpha + (\alpha * xe_{ij})$ is in $N$. Since $N$ is closed under addition, $\alpha * xe_{ij}$ is in $N$.

The fact that $N$ is an abelian ideal follows from the fact that $N$ is an abelian group.

Conversely, assume that $I$ is a maximal abelian ideal of the Lie ring $\mathrm{NT}^*(d,\mathbf{F})$. Then from Lemma 1.2, $I^2 \subseteq I$. This proves that $I$ is a subring, this implies that it is closed under $\circ$. So $I$ is a subgroup, and since $\alpha * xe_{ij} \in I$, for $\alpha \in I$ and $xe_{ij} \in \mathrm{NT}(d,\mathbf{F})$, $\alpha(xe_{ij}) - (xe_{ij})\alpha \in I$. This shows that $I$ is a normal subgroup. •

## 2. Maximal abelian ideals of $\mathrm{NT}^*(d,\mathbf{F})$

Notice that the centralizer of any set in the ring $\mathrm{NT}(d,\mathbf{F})$, the associated group $\mathrm{UT}(d,\mathbf{F})$, and the Lie ring $\mathrm{NT}^*(d,\mathbf{F})$ is identical. Let us look at the centralizer of $\mathbf{N}_{i,j}$. Notice that, if $\left(\sum\limits_{m>n} a_{m,n}e_{m,n}\right) N_{i,j} = N_{i,j}\left(\sum\limits_{m>n} a_{m,n}e_{m,n}\right)$, the left hand side is a linear combination of the rows of $N_{i,j}$ and the right hand side is a linear combination of columns of $N_{i,j}$. Since the entries of $N_{i,j}$ are arbitrary field elements, the only way that this is possible is that $e_{m,n}\mathbf{N}_{i,j} = 0$ and $\mathbf{N}_{i,j}e_{m,n} = 0$ for $m > n$. So to find the centralizer is to look for $m, n$ with $m > n$, such that $e_{m,n}N_{i,j} = 0 = N_{i,j}e_{m,n}$. Now it is easy to see that the centralizer

$$\mathcal{C}\left(\mathbf{N}_{i,j}\right) = \mathbf{N}_{j+1,i-1}.$$

Since $\mathcal{C}\left(\mathbf{N}_{i+1,i}\right) = \mathbf{N}_{i+1,i}$, if $\mathbf{N}_{i+1,i}$ is properly contained in an abelian ideal, then that ideal is contained in the centralizer; which is impossible. This proves that

$$\mathbf{N}_{i+1,i} \tag{6}$$

is a maximal abelian ideal of $\mathrm{NT}^*(d,\mathbf{F})$ for $i = 1, 2, \ldots, d-1$. Further notice that,

$$\Gamma_k = N_{k+1,1} + N_{k+2,2} + \ldots + N_{d,d-k},$$

taking intersection of partition subgroups, it is easy to see that $C\left(\Gamma_k\right) = N_{d-k+1,k}$. In particular, if $d$ is even, i.e., $d = 2k$ for some integer $k$, then $C\left(\Gamma_k\right) = N_{k+1,k}$.

2.1. **Are there any other maximal abelian ideals of $\mathbf{NT}^*(d,\mathbf{F})$?** Let $H$ be a maximal abelian ideal of $\mathrm{NT}^*(d,\mathbf{F})$. Following Levčhuk [1], we define $H_{i,j}$ to be a subset of $\mathbf{F}$, whose elements are in the $(i,j)$ position of a matrix belonging to $H$.

Let $m$ be the smallest, and $n$ be the largest integer such that $H_{m,1}\neq 0$ and $H_{d,n}\neq 0$. Since $H$ is an ideal, for $i>j>1$ and $i<m$, $H*e_{j,1}\in H$. This implies that $H_{i,j}e_{i,1}\in H$. Now for $i<m$, $H_{i,1}=0$, hence $H_{i,j}=0$ for $i<m$.

Similarly, one can show that $H_{i,j}=0$ for $j>n$ by looking at the fact, $H*e_{d,j}\in H$.

Two things can happen, either $n<m$ or $n\geq m$. In the first case, it is clear that $H$ is contained in $\mathbf{N}_{m,m-1}$ and is thus $\mathbf{N}_{m,m-1}$.

If we assume that $n\geq m$, then the description of $H$ is bit involved. It gives rise to maximal abelian ideals of the *e ceptional* type.

First notice, if $n=m$, then $H_{m,j}=0$ for $j>1$ and $H_{i,n}=0$ for $i<d$. This follows from the fact that $H^2\subseteq\mathbf{N}_{d,1}$ (see Lemma 1.2).

In this case $(n=m)$, let $\alpha,\beta\in H$. Then $\alpha*\beta=0$. If we write $\alpha=\sum\limits_{i>j}\alpha_{i,j}e_{i,j}$ and $\beta=\sum\limits_{i>j}\beta_{i,j}e_{i,j}$, then $\alpha_{n,1}\beta_{d,n}-\beta_{n,1}\alpha_{d,n}=0$. Now notice that $H$ being closed under addition, we can assume that $\alpha_{n,1}$ and $\alpha_{d,n}$ are non-zero. This implies that the maximal abelian ideal $H$ is of the form: for a fixed $c\in\mathbf{F}$

$$\left\{\mathbf{N}_{i+1,i-1}+xe_{i,1}+xce_{d,i};\ \ x\in\mathbf{F}\right\}. \tag{7}$$

Now let us consider the case of $n>m$, in this case we first show that $n>m+1$ is impossible.

For $n>m$, we show that $H_{m,i}=0$ for $i>1$. Notice that for $n>m>i>1$, $e_{n,m}*(H*e_{i,1})\in H$. Since $H*H=0$, this implies that $H_{d,n}H_{m,i}e_{d,1}=0$. Since $H_{d,n}\neq 0$, $H_{m,i}=0$ for $i>1$.

In a similar way, looking at $e_{d,j}*(H*e_{n,m})\in H$ shows us that $H_{j,n}=0$ for $j<d$.

Then for $n>m+1$, $(H*e_{n,m+1})*e_{m+1,m}\in H$. This implies that $H_{d,n}e_{d,n}\in H$. The fact that $H*H=0$, gives us that $H_{d,n}H_{m,1}e_{d,1}=0$. However this is impossible. Hence $n>m+1$ is impossible.

Now we show that, if $n=m+1$, then $2\mathbf{F}=0$, i.e., $\mathbf{F}$ has characteristic 2. Since $H_{m,1}$ and $H_{d,m+1}$ are both non-zero. Since $H$ is closed under addition, there is a matrix $\alpha\in H$, where $\alpha=\sum\limits_{i>j}\alpha_{i,j}e_{i,j}$ and $\alpha_{m,1}\neq 0$ and $\alpha_{d,m+1}\neq 0$.

From Lemma 1.2, we know that $2\alpha(xe_{m+1,m})\alpha=0$ for any $x\in\mathbf{F}$, this implies that $2x\left(\alpha_{d,m+1}\alpha_{m,1}e_{d,j}\right)=0$. Since $x$ is arbitrary, we have $2\mathbf{F}=0$.

From Lemma 1.2, $H^2\subset\mathbf{N}_{d,1}$, this implies, $H_{m+1,i}=0$ for $i>1$ and $H_{j,m}=0$ for $j<d$. Let $\alpha=\sum\limits_{i>j}\alpha_{i,j}e_{i,j}$ is in $H$. Since $H$ is closed under addition, we may assume that $\alpha_{m,1},\alpha_{m+1,1},\alpha_{d,m}$ and $\alpha_{d,m+1}$ are all nonzero. Then $(H*e_{m+1,m})*H=0$, implies that $(\alpha*e_{m+1,m})*\beta=0$, where $\beta=$

$\sum_{i>j}\beta_{i,j}e_{i,j}$. This is the same as saying that $\alpha_{d,m+1}\beta_{m,1}+\alpha_{m,1}\beta_{d,m+1}=0$. This implies that there is a $c\in\mathbf{F}^{\times}$ such that $\alpha_{d,m+1}=c\alpha_{m,1}$.

Let

$$\begin{aligned}\alpha = &\quad xe_{i,1}+\alpha_{i+1,1}e_{i+1,1}+\alpha_{d,i}e_{d,i}+cxe_{d,i+1}\\ \beta = &\quad ye_{i,1}+\beta_{i+1,1}e_{i+1,1}+\beta_{d,i}e_{d,i}+cye_{d,i+1}\end{aligned}$$

be two elements of $H$, where $x,y\neq 0$. Since $H$ is an abelian ideal of $\mathrm{NT}^*(d,\mathbf{F})$, $\alpha*\beta=0$. This implies

$$y\alpha_{d,i}+cx\beta_{i+1,1}=x\beta_{d,i}+cy\alpha_{i+1,1}.$$

Consider the substitution

$$\begin{aligned}&x=x_1x,\quad \alpha_{i+1,1}=x_1\alpha_{i+1,1}+y_1,\quad \alpha_{d,i}=x_1\alpha_{d,i}+cy_1;\\ &y=x_2y,\quad \beta_{i+1,1}=x_2\beta_{i+1,1}+y_2,\quad \beta_{d,i}=x_2\beta_{d,i}+cy_2;\end{aligned}$$

where $x_i,y_i\in\mathbf{F}$, $i=1,2$. It is easy to verify, using direct computation, that $\alpha*\beta=0$ if and only if $\alpha'*\beta'=0$ where $\alpha'$ and $\beta'$ are obtained from $\alpha$ and $\beta$ using the above substitutions.

Also, it is easy to verify that if $\alpha\in H$, then $xx_1e_{i,1}+(x_1\alpha_{i+1,1}+y_1)e_{i+1,1}+(x_1\alpha_{d,i}+cy_1)e_{d,i}+cxx_1e_{d,i+1}\in H$. This proves that
$H\subseteq\mathbf{N}_{i+2,i-1}+\{xe_{i,1}+(xa+y)e_{i+1,1}+(xb+cy)e_{d,i}+xce_{d,i+1}\}$, $a,b\in\mathbf{F}$.

Then the maximal abelian ideal $H$ is of the form: for $a,b,c\in\mathbf{F}$, $c\neq 0$ and $i=2,3,\ldots,d-2$,

$$\mathbf{N}_{i+2,i-1}+(xa+y)e_{i+1,1}+(xb+cy)e_{d,i}+xe_{i,1}+cxe_{d,i+1},\quad x\in\mathbf{F}. \tag{8}$$

So by now we have proved a theorem.

**Theorem 2.1** (Weir 1955; Levchuk 1976)**.** The maximal abelian ideals of the Lie ring $\mathrm{NT}^*(d,\mathbf{F})$ are of the following form: (6), (7) and (8) above. The (8) occurs only when the field is of characteristic 2.

## 3. The automorphism group of $\mathrm{UT}(d,\mathbf{F})$

In this section we describe the generators of the automorphism group of the group $\mathrm{UT}(d,\mathbf{F})$. These automorphisms being well-known, we just present them as action on $xe_{i+1,i}$, $i=1,2,\ldots,d-1$ and $x\in\mathbf{F}$, and leave as an exercise the prove that these are automorphisms. The automorphisms are as follows:
**Extremal automorphisms − $\mathbf{Aut}_E$.** These automorphisms arise from the maximal abelian ideals of exceptional type. As we saw, the maximal abelian ideals of exceptional type are different for a field of characteristic 2. So, we will have two different types of automorphisms. One for even characteristic and other for the field of odd characteristic.

**Odd Characteristic**.

$$xe_{2,1} \mapsto xe_{2,1} + axe_{d,2} + x^{\lambda}e_{d,1} \tag{9}$$

Where $x^{\lambda} : \mathbf{F} \to \mathbf{F}$ is a map that satisfies the equation $(x+y)^{\lambda} - x^{\lambda} - y^{\lambda} = axy$ and $a = 2^{\lambda} - 2(1^{\lambda})$. All other generators remain fixed.

Similarly, one can define $xe_{d,d-1} = xe_{d,d-1} + axe_{d-1,1} + x^{\lambda}e_{d,1}$. All other generators remain fixed and the $\lambda$ satisfies the above relations.

If $\mathbf{F}$ is of even characteristic, then $a = 0^{\lambda}$. It is easy to see that, since $0^{\lambda} = 0$, $a = 0$. So in the case of the characteristic of the field to be even, the extremal automorphisms become the central automorphisms.

Clearly automorphisms of this form generate a subgroup of the full automorphism group of $\mathrm{UT}(d, \mathbf{F})$, denoted by $\mathrm{Aut}_E$ and is isomorphic to $\mathbf{F}^+ \oplus \mathbf{F}^+$.

**Even Characteristic**.

$$xe_{2,1} \mapsto xe_{2,1} + axe_{d,3} \tag{10}$$

all other generators remain fixed. Similarly one can define $xe_{d,d-1} \mapsto xe_{d,d-1} + axe_{d-1,1}$. Again this automorphism group $\mathrm{Aut}_E$ is isomorphic to $\mathbf{F}^+ \oplus \mathbf{F}^+$. We will later show that these automorphisms are only possible when the field $\mathbf{F}$ is the field of two elements $\mathbb{Z}_2$.

**Flip automorphism – Aut$_F$**. This automorphism is given by flipping the matrix by the anti-diagonal and is given by the equation:

$$xe_{i,j} \mapsto xe_{d-j+1,d-i+1} \tag{11}$$

This is clearly an automorphism of order 2 and forms a subgroup of the automorphism group and will be denoted by $\mathrm{Aut}_F$.

**Diagonal automorphisms – Aut$_D$**. This automorphism is conjugation by a diagonal matrix. Diagonal matrices are defined as matrices with only non-zero terms in the main diagonal and everything else zero. Let $D = [x_1, x_2, \ldots, x_d]$ be a diagonal matrix, with $x_1, x_2, \ldots, x_d$ as the non-zero diagonal entries in the respective rows. Then $D^{-1}xe_{i,j}D = d_i^{-1}xd_je_{i,j}$. So a diagonal matrix maps $xe_{i,j} \mapsto d_i^{-1}xd_je_{i,j}$. The kernel of this map is the set of all scalar matrices, i.e., $x_1 = x_2 = \cdots = x_d$. This is clearly a subgroup of the automorphism group, which is of the form $\mathbf{F}^{\times} \times \mathbf{F}^{\times} \times \ldots \times \mathbf{F}^{\times}$ ($d-1$ times), and will be denoted as $\mathrm{Aut}_D$.

**Field automorphisms – Aut$_A$**. This automorphisms can be described as

$$xe_{i+1,i} \mapsto x^{\mu}e_{i+1,i} \quad i = 1, 2, \ldots, d-1. \tag{12}$$

Where $\mu : \mathbf{F} \to \mathbf{F}$ is a field automorphism.

**Inner automorphisms – Aut$_I$**. This is the well known normal subgroup of the automorphism group in any non-abelian group; where $x \mapsto g^{-1}xg$ for some $g \in \mathrm{UT}(d, \mathbf{F})$ and $x \in \mathrm{UT}(d, \mathbf{F})$.

**Central automorphisms – $\mathbf{Aut}_C$.** Central automorphisms are the centralizers of the group of inner automorphisms in the group of automorphisms. The simplest way to explain them is to "multiply" the generators with an element of the center. In the case of $\mathrm{UT}(d, \mathbf{F})$ it is

$$xe_{i+1,i} \mapsto xe_{i+1,i} + x^{\lambda} e_{d,1} \tag{13}$$

Where $\lambda$ is a linear map of $\mathbf{F}^+$ to itself.

3.1. **Why are these the generators of the automorphism group of $\mathbf{UT}(d, \mathbf{F})$?** We know that any automorphism $\phi$ of any group maps a maximal abelian normal subgroup to a maximal abelian normal subgroup. Our first lemma uses that to prove:

**Lemma 3.1.** Let $\phi$ be an automorphism of $\mathrm{UT}(d, \mathbf{F})$. Then $\mathbf{N}^{\phi}_{i+1,i}$ is either $\mathbf{N}_{i+1,i}$ or $\mathbf{N}_{d-i+1,d-i}$, for $i = 1, 2, \ldots, d-1$.

*Proof.* Notice that the centralizer $C(\Gamma_k)$ of $\Gamma_k$, the $k^{\text{th}}$ element in the central series is characteristic. If $d$ is even, and $d = 2k$ for some $k$, then $C(\Gamma_k) = N_{k+1,k}$. Hence $N^{\phi}_{k+1,k} = N_{k+1,k}$. Further notice that for $i < \frac{d}{2}$, $\mathbf{N}_{i+1,i} \cap \mathbf{N}_{d-i+1,d-i} = \mathbf{N}_{d-i+1,i}$ is a characteristic subgroup of $\mathrm{UT}(d, \mathbf{F})$. Since flip is an automorphism of $\mathrm{UT}(d, \mathbf{F})$, any characteristic subgroup must be symmetric about the second diagonal.

Then $\mathbf{N}_{d-i+1,i}$ is the maximal characteristic subgroup of $\mathrm{UT}(d, \mathbf{F})$ contained in both $\mathbf{N}_{i+1,i}$ and $\mathbf{N}_{d-i+1,d-i}$. This means that it must be contained as a maximal characteristic subgroup in $\mathbf{N}^{\phi}_{i+1,i}$. So $\mathbf{N}^{\phi}_{i+1,i}$ has two choices, $\mathbf{N}_{i+1,i}$ or $\mathbf{N}_{d-i+1,d-i}$. ∙

It is important to notice here that $xe_{2,1}$ and $xe_{d,d-1}$ are not only contained in the maximal abelian normal subgroups $\mathbf{N}_{2,1}$ and $\mathbf{N}_{d,d-1}$. They are also contained in the exceptional subgroups. Let us call the maximal abelian normal subgroup of type (7) (or of type (8), when characteristic of the field is 2) containing $xe_{2,1}$ as $\mathbf{A}_2$ and the maximal abelian normal subgroup containing $xe_{d,d-1}$ as $\mathbf{A}_{d-1}$.

It is clear from the last lemma, that $\mathbf{A}^{\phi}_2$ is either $\mathbf{A}_2$ or $\mathbf{A}_{d-1}$. Then clearly, if necessary, composing $\phi$ with the flip automorphism, we claim that $\mathrm{UT}(d, \mathbf{F})/\Gamma_2$ is invariant under $\phi$. We can actually say more, $xe^{\phi}_{i+1,i} = x^{\lambda_i} e_{i+1,i} \mod \Gamma_2$, for $i = 1, 2, \ldots, d-1$. Where $\lambda_i : \mathbf{F} \to \mathbf{F}$ is a map. Now let us try to understand the map $\lambda_i$. Since $\phi$ is an automorphism, each map $\lambda_i$ is a bijection.

Now recall the relations in $\mathrm{UT}(d, \mathbf{F})$ (Equations 2). It follows from the relation $xe_{i,j} \circ ye_{i,j} = (x+y)e_{i,j}$, if $xe_{i+1,i} \mapsto x^{\lambda_i} e_{i+1,i}$ then $\lambda_i$ is a linear map of $\mathbf{F}^+$.

Furthermore, since $[xe_{i+1,i}, ye_{i,i-1}] = [ye_{i+1,i}, xe_{i,i-1}]$, $x^{\lambda_i} y^{\lambda_{i-1}} = y^{\lambda_i} x^{\lambda_{i-1}}$, for $i = 2, 3, \ldots, d-1$. Also, since $[xe_{2,1}, ye_{3,2}]^{\phi} = [ye_{2,1}, xe_{3,2}]$, $x^{\lambda_1} y^{\lambda_2} = y^{\lambda_1} x^{\lambda_2}$.

Taking $y = 1$ above, it follows that $\lambda_1 = k_1\lambda_2 = k_2\lambda_3 = \cdots = k_{d-2}\lambda_{d-1}$. Where $k_i$ are nonzero fields elements.

Recall that conjugating by a diagonal matrix multiplies the first subdiagonal by scalars. Then one can conjugate by a suitable diagonal matrix, to find that $\lambda_i = \lambda$ for all $i$. Then the commutator relations show that $\lambda$ is closed under multiplication. So now we are in a position to claim, that composing $\phi$ with a field automorphism and a diagonal automorphism, $\phi$ maps like the identity on $\mathrm{UT}(d, \mathbf{F})/\Gamma_2$.

As we saw from the above lemma, $\phi$, (after composing with the flip, if necessary) maps $\mathbf{A}_2$ and $\mathbf{A}_{d-1}$ to itself. So it follows that $xe_{2,1}^{\phi}$ and $xe_{d,d-1}^{\phi}$ are in $\mathbf{A}_2$ and $\mathbf{A}_{d-1}$ respectively. In case of odd characteristic, the description of the extremal automorphism (Equation 9) implies, we can compose $\phi$ with an extremal automorphisms, such that, $xe_{2,1}^{\phi}$ and $xe_{d,d-1}^{\phi}$ are in $\mathbf{N}_{2,1}$ and $\mathbf{N}_{d,d-1}$ respectively.

In case of the even characteristic we need to say more. Notice that in the case that the characteristic of the field $\mathbf{F}$ being even, the maximal abelian normal subgroup containing $e_{2,1}$ is

$$\mathbf{N}_{4,1} + ae_{3,1} + be_{d,2} + xe_{2,1} + cxe_{d,3}$$

where $a, b \in \mathbf{F}$, $c \in \mathbf{F}^{\times}$, $x \in \mathbf{F}$. We want to know more about the automorphism that moves $e_{2,1}$. Using the flip automorphism, if necessary, we can assume that $\mathbf{A}_2^{\phi} = \mathbf{A}_2$. So the only choice is

$$xe_{2,1} \mapsto xe_{2,1} + axe_{d,3} + x^{\lambda}e_{d,2} + x^{\mu}e_{d,1}$$

where $\lambda, \mu : \mathbf{F} \to \mathbf{F}$. From relation 2, we see that $(x+y)^{\lambda} = x^{\lambda} + y^{\lambda}$. This implies that $0^{\lambda} = 0$. Also it follows that $(x+y)^{\mu} = x^{\mu} + y^{\mu} + x^{\lambda}y$. This implies that $(x+y)^{\mu} = x^{\mu} + y^{\mu} + 1^{\lambda}xy$. Putting $x = 0$ and $y = 1$, we have that $0^{\mu} = 0$. Putting $x = 1$ and $y = 1$ we get $1^{\lambda} = 0$. Since $x^{\lambda} = 1^{\lambda}x$, this implies that $x^{\lambda} = 0$, i.e., $\lambda$ is the zero map.

Once $\lambda$ is the zero map, clearly the $e_{d,1}$ entry doesn't matter. Hence using Equation 10, composing $\phi$ with extremal automorphisms, one can show that $xe_{2,1}^{\phi}$ and $xe_{d,d-1}^{\phi}$ are in $\mathbf{N}_{2,1}$ and $\mathbf{N}_{d,d-1}$ respectively.

From the commutator relations (Relation 3), we see that the

$$xye_{3,1} \mapsto xye_{3,1} + ax^2ye_{d,1} + axye_{d,2}.$$

Then interchanging $x$ and $y$, we see that $axy(y-x) = 0$. Since $a \neq 0$, so in the field $\mathbf{F}$, for any two distinct element, one is zero. This means that $\mathbf{F} = \mathbb{Z}_2$.

Notice that for $i = s$, $[xe_{i+1,i}, ye_{s,t}] = xye_{i+1,t}$. So if $s - t = k$, then $(i+1)-t = k+1$. Using this idea one can clear each and every subdiagonals, one after another, starting with $k = 2$. This means that by suitable conjugation, $xe_{i+1,i}^{\phi}$ will have no non-zero entries except the $(d,1)$ entry. In the case of $xe_{2,1}^{\phi}$ and $xe_{d,d-1}^{\phi}$ one can actually clear all non-zero entries, including the $(d,1)$ entry, using conjugation. One can choose the conjugators in such a way that this gives rise to an inner automorphism. This proves the following lemma:

**Lemma 3.2.** Let $\Gamma_k$ be as defined. For an automorphism $\phi$ of $\mathrm{UT}(d, \mathbf{F})$, which fixes $\mathrm{UT}(d, \mathbf{F}) \mod \Gamma_2$, one can use inner automorphisms, such that $\phi$ acts like the identity modulo $\Gamma_{d-1}$.

So now we have the following:

- Use the flip automorphism, if necessary, so that $\mathbf{N}_{i+1,i}^{\phi} = \mathbf{N}_{i+1,i}$ for $i = 2, 3, \ldots, d-2$.
- Use extremal automorphisms, if necessary, so that $\mathbf{N}_{i+1,i}^{\phi} = \mathbf{N}_{i+1,i}$ for $i = 1, d-1$.
- Use a field automorphism and a diagonal automorphism, if necessary, so that $xe_{i+1,i}^{\phi} = xe_{i+1,i} \mod \Gamma_2$.
- Use inner automorphisms, if necessary, so that $xe_{i+1,i}^{\phi} = xe_{i+1,i} \mod \Gamma_{d-1}$.
- Use central automorphisms, if necessary, so that $xe_{i+1,i}^{\phi} = xe_{i+1,i}$. Note that the central automorphisms corresponding to $xe_{2,1}$ and $xe_{d,d-1}$ are inner automorphisms.

Now we have proved the following theorem:

**Theorem 3.3.** The automorphism group of $\mathrm{UT}(d, \mathbf{F})$ is generated by extremal automorphisms, field automorphisms, diagonal automorphisms, inner automorphisms and central automorphisms.

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