

## Some Commutativity Results for Certain Rings

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### Abstract

The aim of the present paper is to investigate the commutativity of a ring with unity satisfying one of the following properties:

$$\{1 - p(yx^m)\}[yx^m - yx^m b(x^m y), x]^r \{1 - q(yx^m)\} = 0$$

$$\text{and } \{1 - \tilde{p}(xy^m)\}[y^m x \tilde{b}(xy^m), y] \{1 - \tilde{q}(xy^m)\} = 0, \quad \text{For some } x, y \in R$$

$$y^r [x^n, y] = g(x)[x^q f(x), y]^r h(x) \text{ or } [x^n, y]^r y^r = \tilde{g}(x)[x^q \tilde{f}(x), y] h(x),$$

$$b(X), \tilde{b}(X) \text{ in } X^{-2} Z[X] \text{ and } p(X), \tilde{p}(X), q(X), \tilde{q}(X) \in Z[X]$$

and  $f(X), \tilde{f}(X), g(X), \tilde{g}(X), h(X), h(X)$  in  $Z[X]$ ; where  $m \geq 0, r \geq 0, q \geq 2, n > 0$  are integers. Also we extend these results to the case when integral exponents in the underlying conditions are no longer fixed, rather they depend on the pair of ring elements  $x, y$  for their values. Finally under different appropriate constraints on commutators, commutativity of  $s$ -unital rings has been discussed.

**Mathematics Subject Classifications:** 16U80

**Keywords:** Division ring, factor subring, polynomial identity,  $s$ -unital ring, torsion-free ring

## 1. Introduction

Throughout,  $R$  will represent an associative ring (may be without unity),  $N = N(R)$ , the set of nilpotent elements of  $R$ ,  $Z = Z(R)$ , the center of  $R$ ,  $C = C(R)$ , the commutator ideal of  $m \geq 0$  and  $U = U(R)$ , the group of units of  $X$ . For any  $x, y \in R$ ,  $[x, y]$  denotes the commutator  $xy - yx$ . As usual  $Z[X]$  is the totality of polynomials in  $X$  with coefficients in  $Z$ , the ring of integers. Consider the following ring properties:

(I). For each  $x, y \in R$ ; there exist an integer  $m \geq 0$  and the polynomials

$$b(X), \tilde{b}(X) \in X^2Z[X] \text{ and}$$

$$p(X), \tilde{p}(X), q(X), \tilde{q}(X) \in Z[X] \text{ such that}$$

$$\{1 - p(yx^m)\}[yx^m - yx^m b(x^m y), x]\{1 - q(yx^m)\} = 0$$

$$\text{and } \{1 - \tilde{p}(xy^m)\}[y^m x - y^m x \tilde{b}(xy^m), y]\{1 - \tilde{q}(xy^m)\} = 0, \text{ where } m \geq 0 \text{ is a fixed integer.}$$

(II). For each  $x, y \in R$  there is an integer  $m \geq 0$  and the polynomials  $b(X), \tilde{b}(X) \in X^2Z[X]$  and

$$p(X), \tilde{p}(X), q(X), \tilde{q}(X) \in Z[X]$$

$$\text{such that } \{1 - p(yx^m)\}[yx^m - yx^m b(x^m y), x]\{1 - q(yx^m)\} = 0$$

$$\text{and } \{1 - \tilde{p}(xy^m)\}[y^m x - y^m x \tilde{b}(xy^m), y]\{1 - \tilde{q}(xy^m)\} = 0$$

(CH). For each  $x, y \in R$  there exist  $b(t), g(t) \in t^2Z[t]$  such that  $[x - g(x), y - b(y)] = 0$ .

A beautiful theorem of Herstein [3] asserts that if  $R$  is a ring satisfying the property (CH), then  $R$  is commutative. It is natural to consider the related properties;  $[xy - p(xy), x] = 0$  and  $[xy - q(yx), x] = 0$  for some  $p(X), q(X)$  in  $X^2Z[X]$  depending on the ring elements  $x, y$ .

Putchá and Yaqub [9] remarked that if for every  $x, y \in R$ , there exists a polynomial  $p(X) \in X^2Z[X]$  such that  $xy - p(xy)$  is central, then  $R^2$  must be central. Motivated by these observations, the author [5] found the commutativity of rings with unity 1 satisfying the property  $[yx^m - x^n b(y)x^l, x] = 0$ , where the polynomial  $b(x)$  in  $X^2Z[x]$  depends on the pairs

$x, y \in R$  and fixed non-negative integers  $l, m, n$ . Hence now one can ask, what can we say about the commutativity of ring  $R$ , if the underlying condition is replaced by  $[yx^m - x^n b(y)x^l, x] = 0$ ? We prove rather a more general result by establishing that a ring with 1 satisfying the property (I) is commutative. Further we shall consider the property (II), where integral exponents are allowed to vary with the pair of ring elements  $x, y$  and also the ring satisfies the Chacron's condition (CH). Finally, in Section 4, we establish commutativity of s-unital rings satisfying the property (see [5]) under appropriate torsion restrictions on commutators. Also several commutativity results can be obtained as corollaries to our results (see [4, 5, 8, 10, 12]).

## 2. Results

Consider the following types of rings.

$$(i) \begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}, p \text{ a prime}$$

$$(i) \begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}, p \text{ a prime.}$$

$$(i) \begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}, p \text{ a prime.}$$

$$(ii) m_\sigma(F) = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \mid a, b \in F \right\}, \text{ where } F \text{ is a finite with a non-trivial automorphism } \sigma.$$

(iii) A non-commutative division ring.

(iv)  $S = \langle 1 \rangle + T$ ,  $T$  a non-commutative radical subring of  $S$ , must be a domain.

(v)  $S = \langle 1 \rangle + T$ ,  $T$  a non-commutative subring of  $S$  that  $T[T, T] = [T, T]T = 0$ .

In 1989, Streb [12] gave a nice classification for non-commutative rings which yields a

powerful tool in obtaining a number of commutativity theorems ([5], [6], and [7]). It follows from the proof of [6, Corollary 1] that if  $R$  is a non-commutative ring with unity 1, then there exists a factorsubring of  $R$  which is of type (i), (ii), (iii), (iv) or (v). This observation gives the following proposition that plays a vital role in our subsequent discussion.

**Proposition 2.1.** Let  $p$  be a ring property which is inherited by factor subrings. If no ring of type (i), (ii), (iii), (iv), or (v) satisfies  $P$  then every ring with unity 1 and satisfying  $P$  is commutative.

We state the following known results.

**Lemma 2.1 [3]** Let  $f$  be a polynomial in  $n$  non-commuting indeterminates  $x_1, x_2, \dots, x_n$  with relatively prime integral coefficients. Then the following are equivalent.

- (a) For any ring  $R$  satisfying the polynomial identity  $f = 0$ ,  $\mathbf{C}$  is a nil ideal.
- (b) For every prime  $p$ ,  $(GF(p))_2$  fails to satisfy  $f = 0$ .
- (c) Every semiprime ring satisfying  $f = 0$  is commutative.

**Lemma 2.2 [7]** If  $R$  is non commutative ring satisfying (CH), then there exists a factor subring of  $R$  which is of type (i), or (ii).

**Lemma 2.3 [1]** Let  $R$  be a ring in which for all  $x, y$  in  $R$ , there exists polynomial  $f(X)$  in  $X^2\mathbf{Z}[X]$  such that  $[x - f(x), y] = 0$ . Then  $R$  is commutative.

### 3. Commutativity of rings with unity 1

**Theorem 3.1** Let  $R$  be a ring with unity 1 satisfying (I). Then  $R$  is a commutative ring.

**Proof.** In view of Proposition 2.1, it is enough to prove that  $R$  cannot be of type (i), (ii), (iii), (iv), (v).

Choose some polynomials  $b(X) \in X^2Z[X]$  and  $p(X), q(X)$  in  $XZ[X]$  such that

$$\{1 - p(yx^m)\}[yx^m - yx^m b(x^m y), x]\{1 - q(yx^m)\} = 0.$$

Taking  $w \in R$  such that  $b(x^m y) = x^m w$  and  $b(yx^m) = wx^m$

For some  $b(X)$  in  $X^2Z[X]$  and  $\tilde{p}(X), \tilde{q}(X), \tilde{b}(X)$  in  $XZ[X]$  we get

$$(3.1) \quad \{1 - \tilde{p}(b(yx^m))\}[b(x^m y) - b(yx^m) \tilde{b}(b(yx^m)), x]\{1 - \tilde{q}(b(yx^m))\} = 0$$

and

$$(3.2) \quad \{1 - \tilde{p}(wx^m)\}[x^m w - wx^m \tilde{b}(wx^m), x]\{1 - \tilde{q}(wx^m)\} = 0$$

Combining (3.1) and (3.2), we get

$$\{1 - \tilde{p}(b(yx^m))\}\{1 - p(yx^m)\}[yx^m - yx^m \tilde{b}(b(yx^m)), x]\{1 - q(yx^m)\}\{1 - \tilde{q}(b(yx^m))\} = 0$$

This implies that

$$(3.3) \quad \{1 - p(yx^m)\}[yx^m - yx^m b(yx^m), x]\{1 - q(yx^m)\} = 0$$

Let  $R$  be a ring of type (i) suppose that  $R$  satisfies the property (I). Then  $R$  satisfies (3.3).

Taking  $x = e_{11}$  and  $y = e_{21}$ , we find

$$\{1 - p(e_{21}e_{11}^m)\}[e_{11}, e_{21}e_{11}^m - e_{21}e_{11}^m b(e_{21}e_{11}^m)]\{1 - q(e_{21}e_{11}^m)\} = e_{21} \neq 0, \text{ a contradiction.}$$

Let the ring  $R = M_\sigma(F)$  be of type (ii). If  $R$  satisfies the property (I). Then choose

$$\begin{pmatrix} a & 0 \\ 0 & 0(a) \end{pmatrix}, (\alpha \neq \sigma(\alpha)) \text{ and } y = e_{21} \text{ we get}$$

$$\{1 - p(yx^m)\}[x, yx^m - yx^m b(yx^m)]\{1 - q(yx^m)\} = e_{12}(\alpha - \sigma(\alpha)) \neq 0$$

Thus  $R$  cannot be of type (ii).

Let  $R$  be a ring of type (iii). Suppose that  $R$  satisfies the property (I). Let  $u$  be a unit in  $R$ , that is  $u \in U$ , and for arbitrary element  $y \in R$  we obtain polynomials

$b(X) \in X^2Z[X]$  and  $p(X), q(X) \in XZ[X]$  such that

$$\{1 - p(yu^{-m}u^m)\}[yu^{-m}u^m - yu^{-m}u^m b(yu^{-m}u^m), u]\{1 - q(yu^{-m}u^m)\} = 0$$

This implies that

$$\{1 - p(y)\}[y - yb(y), u]\{1 - q(y)\} = 0.$$

This shows that either  $1 - g(y) = 0, 1 - h(y) = 0$ , or  $[y - yf(y), u] = 0$ .

In all the cases,  $R$  is commutative by Lemma 2:3, a contradiction.

Let  $R$  be of type (iv). Let  $R$  satisfy (I). Then a careful scrutiny of the proof of type (iii) gives that there exist  $x$  and  $u \in U$  and arbitrary  $y \in R$  such that either  $[x, y - b(y)] = 0, y - yp(y) = 0$  or  $y - yq(y) = 0$  for all  $b(X) \in X^2Z[X]$  and  $p(X), q(X) \in XZ[X]$ . But in the present case if  $t_1, t_2 \in T$  then  $u = 1 + t_1$  is a unit and there exist  $b(X) \in X^2Z[X]$  and  $p(X), q(X) \in XZ[X]$  such that either  $[t_1 - q(t_2), 1 + t_1] = 0, t_2 - t_2 p(t_2) = 0$  or  $t_2 - t_2 q(t_2) = 0$ . Thus, in every case  $T$  is commutative by Lemma 2.3, a contradiction.

Further, let  $R$  be of type (v). Let  $t_1, t_2 \in T$  such that  $[t_1, t_2] \neq 0$  suppose that  $R$  satisfies (I). Then there exist polynomials  $b(X) \in X^2Z[X]$  and  $p(X), q(X) \in XZ[X]$  such that

$$\{1 - p(t_2(1+t_1)^m)\}[1+t_1, t_2(1+t_1)^m - t_2(1+t_1)^m b(t_2(1+t_1)^m)]\{1 - q(t_2(1+t_1)^m)\} = 0$$

This implies that  $[t_1, t_2] = 0$ , a contradiction.

Hence, one can see that no ring of type (i), (ii), (iii), (iv) or (v) satisfies (I) and an application of Proposition 2.1,  $R$  is commutative.

**Corollary 3.1** Let  $l, m, n$  be fixed non-negative integers and  $R$  be a ring with unity 1. If for each  $x, y \in R$ , there exists a polynomial  $b(X)$  in  $X^2Z[X]$  such that  $[yx^m - x^n b(y)x^l, x] = 0$  then  $R$  is commutative.

**Remark 3.1** Given an integral exponent  $m$  with the property (I) which is allowed to vary with the pair of elements  $x$  and  $y$ , that is, if  $R$  satisfies the property (II), then a careful scrutiny of the proof of theorem 3.1 asserts that  $R$  has no factor subring of type (i) or (ii). Further, if  $R$  satisfies the property (CH), then in view of Lemma 2.3, we get the following.

**Theorem 3.2** Suppose that  $R$  is a ring with unity 1 satisfying (CH). Moreover, if  $R$  satisfies the property (II) then  $R$  is commutative (and conversely).

#### 4. Commutativity of torsion free s-unital rings

Since there are non commutative ring with  $R^2$  being central, neither of these conditions guarantees the commutativity in arbitrary rings. Following [3], a ring  $R$  is called left (resp. right) s-unital ring if  $x \in Rx$  (resp.  $x \in xR$ ). A ring  $R$  is called s-unital if and only if  $x \in xR \cap Rx$  for all  $x \in R$ . If  $R$  is s-unital (resp. left or right s-unital), then for any finite subset  $F$  of  $R$  there exists an element  $e \in R$  such that  $ex = xe = x$  (resp.  $ex = x$  or  $xe = x$ ) for all  $x \in F$ . Such an element  $e$  will be called a pseudo (resp. a pseudo left or pseudo right) identity of  $F$  in  $R$ .

**Remark 3.1** The following example demonstrates that in the Theorems 3.1 and 3.2 of [5], the existence of both the conditions is not superfluous (even if ring  $R$  has unity 1).

**Example 4.1** Consider  $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in GF(2) \right\}$ . Then  $R$  is a non-commutative

ring with unity satisfying the condition  $y^r[x, y^4] = x^r[x^4, y]x^8$  where  $r, s$  and  $t$  may be any non-negative integers.

In view of Example 3.1, one may ask a natural question: what additional conditions are needed to establish the commutativity of an arbitrary ring  $R$  if we simply assume

$y^r[x, y^m] = g(x)[x^q f(x), y]h(x)$  and  $[x, y^m]y^r = \tilde{g}(x)[x^q \tilde{f}(x), y]\tilde{h}(x)$  in [5, Theorems 3.1 and 3.2]. Naturally, it is tempting to conjecture that an  $m$ -torsion free ring with unity 1 satisfying any one of the above properties must be commutative. Thus, under certain appropriate constraints on the commutators involved in the underlying conditions we can prove some interesting cases of the conjecture. In fact, we shall consider the following properties.

(III) For each  $x, y \in R$ , there exist polynomials  $f(X), g(X), h(X)$  in  $Z[X]$  such that

$$[x^n, y^r] = g(x)[x^q f(x), y]h(x) \text{ Or } [x^n, y^r] = \tilde{g}(x)[x^q \tilde{f}(x), y]\tilde{h}(x), \text{ where } r > 1, q \geq 2, n \geq 1 \text{ are fixed integers.}$$

(IV) For every  $x, y \in R$ , there exist polynomials  $f(X), g(X), h(X)$  in  $Z[X]$  such that

$$y^r[x^m, y] = g(x)[x^q f(x), y]h(x) \text{ or } [x^m, y]y^r = \tilde{g}(x)[x^q \tilde{f}(x), y]\tilde{h}(x), \text{ where } r > 1, q \geq 2, n \geq 1 \text{ are fixed integers.}$$

To prove the commutativity of ring  $R$  with the above properties we need some extra condition on commutation in  $R$  such as property:  $Q(m)$  For any  $x, y \in R, m[x, y] = 0$  where  $m$  is some positive integer.

An iteration technique was developed by Tong [14]. The method of Tong's proof originates the following generalization of the commutative law:  $xy^m x^m y = x^{m+1} y^{n-1}$ .

Indeed, we begin with



**Lemma 4.1 [14].** Let  $R$  be a ring with 1, and  $I_0^k(x) = x^k$  if  $p \geq 1$ , let  
 $I_p^k(x) = I_{p-1}^k(x+1) - I_{p-1}^k(x)$

For all  $x$  in  $R$  then  $I_{k-1}^k(x) = \frac{1}{2}(k-1)k! + k!x$ ;

$$I_k^k = k! \text{ and } I_j^k(x) = 0 \text{ for } j > k.$$

**Lemma 4.2 [14]** Let  $R$  be a s-unital ring and let  $K$  be a finite subset of  $R$  such that  $f = f(K) \in R$ , a pseudo identity of  $K$ . Let  $t$  be a positive integer. For all  $x \in K$  define  $I_p^q$  recursively as  $I_0^q(x) = x^q$ , and  $I_p^q(x) = I_{p-1}^q(f^t + x) - I_{p-1}^q(x)$ , if  $p \geq 1$ .

Further if  $f^t = f$ , then

$$(4.1) \quad I_{q-1}^q(x) = \frac{1}{2}(q-1)q!f + q!x \quad I_p^q = q!f \text{ and } I_s^q(x) = 0 \text{ for } s > q$$

Let  $f^t \neq f$ . Then

$$(4.2) \quad I_{q-1}^q(x) = f^{t_1} + \dots + f^{\frac{t_{(q-1)q!}}{2}} + q!xI_q^q = q!f^t \text{ and } I_s^q(x) = 0 \text{ for } s > q$$

Our objective is to extend the properties (III) and (IV) to s-unital rings by using the iteration technique.

Now, we are in a position to prove the following theorem.

**Theorem 4.1** Let  $R$  be s-unital ring satisfying any of the properties (III) and (IV). Moreover, if  $R$  satisfies  $Q((\max\{r, n\})!)$ , then  $R$  is commutative.

**Proof.** Let  $R$  satisfy (III). Then first we shall use induction on  $y^r$ . From the Lemma 4.2 we have  $I_k(x) = I_k^r(x)$  for  $k = 0, 1, 2, 3, 4, 5, \dots$

Then condition (III) can be written as

$$(4.3) \quad [x^n, I_0(y)] = g(x)[f(x), y]h(x)$$

Let  $e \in R$  be a pseudo identity of  $\{x, y\} \subseteq R$  and let  $t$  be a non-negative integer. Clearly  $[x, y] = [x, e^t + y] = [e^t + x, y]$ . Replace  $y$  by  $(e^t + y)$  in 4.3 and using Lemma we get

$$[x^n, I_0(y) + I_1(y)] = g(x)[x^q f(x), y]h(x)$$

Using 4.3 we get

$$(4.4) \quad [x^n, I_0(y)] = 0 \text{ for all } x, y \text{ in } R$$

Putting  $y$  by  $(e^t + x)$  and using Lemma 4.2 we get

$$[x^n, I_1(y+1)] = [x^n, I_1(y) + I_2(y)] = 0$$

Again using (4.4) we get  $[x^n, I_2(y)] = 0$ . Replacing  $y$  by  $(e^t + x)$  and then iterating  $(r-1)$  times, we get

$$(4.5) \quad [x^n, I_{r-1}(y)] = 0$$

By Lemma 4.2, equation 4.5 gives  $r![x^n, y] = 0$

Finally, replacing  $x$  by  $(e' + x)$  and using the similar techniques as above we obtain  $r!n![x, y] = 0$  and hence by the property  $Q((\max\{r, n\})!)$ , yields the commutativity of  $R$ . Let  $R$  satisfy (IV). Then using the iteration technique as above, we get

$$I_0(y)[x^n, y] = g(x)[x^q f(x), y]h(x) \text{ Or } [x^n, y]I_0(y) = g(x)[x^q f(x), y]h(x).$$

Replacing  $y$  by  $(e' + x)$  and using Lemma 4.2 we obtain  $I_1(y)[x^n, y] = 0$  or  $[x^n, y]I_1(y) = 0$

Proceeding in the same line as above, we finally get  $I_r(y)[x^n, y] = 0$  or  $[x^n, y]I_r(y) = 0$

Thus in both the cases we get  $r![x^n, y] = 0$

Using the same way of replacing  $x$  by  $(e' + x)$  and iterating  $(n-1)$  times we get  $r![x^n, y] = 0$ , and by the property  $Q((\max\{r, n\})!)$  yields the commutativity of  $R$ .

We conclude our discussion with the following conjecture.

**Conjecture:** Let  $m > 1, r \geq 0, s \geq 0$  be fixed non-negative integers and  $R$  a ring with unity 1 in which for each  $x$  in  $R$  there exist a polynomial  $f(X, Y) = f_x(X, Y)$  in  $R \langle X, Y \rangle$  satisfying the condition that for all  $y$  in  $R, f(x, y) = f(x, y+1) = f(x, x+y)$  so that the property for all  $y$  in  $R$ . Then  $R$  is commutative if it satisfies the property  $Q(m)$ .

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