International Journal of Algebra, Vol. 6, 2012, no. 9, 427 - 432

Rational Gottlieb Group of Function Spaces of Maps into an Even Sphere

J.-B. Gatsinzi[1]

Department of Mathematics, University of Botswana
Private Bag 0022 Gaborone, Botswana
gatsinzj@mopipi.ub.bw

Abstract. Let X be a simply-connected space and $f : X \to S^{2n}$ a mapping into the even sphere. We show that the dimension of the rational Gottlieb group of the universal cover $\widetilde{\mathrm{map}}(X, S^{2n}; f)$ of the function space $\mathrm{map}(X, S^{2n}; f)$ is at least equal to the dimension of $\tilde{H}^*(X, \mathbb{Q})$, if the image of $\tilde{H}^*(f, \mathbb{Q})$ is in the annihilator of $\tilde{H}^*(X, \mathbb{Q})$.

Mathematics Subject Classification: Primary 55P62, 54C35

1. Introduction

An element $\alpha \in \pi_n(X)$ is called a Gottlieb element of X if the map $(\alpha, id) : S^n \vee X \to X$ extends to $\tilde{\alpha} : S^n \times X \to X$ [7]. Gottlieb elements form a subgroup $G_n(X)$ of $\pi_n(X)$. Moreover $G_n(X)$ is the image of the connecting map of the long exact sequence of homotopy groups of the universal fibration $X \to B\,\mathrm{aut}^\bullet X \to B\,\mathrm{aut}\, X$ [8]. Here $\mathrm{aut}\, X$ denotes the monoid of self homotopy equivalences of X and $\mathrm{aut}^\bullet X$ its submonoid of pointed self homotopy equivalences. Therefore, given a fibration $X \to E \to B$, the image of the connecting map $\delta : \pi_{*+1}(B) \to \pi_*(X)$ is contained in the Gottlieb group $G_*(X)$. Moreover, if X_0 denotes the rationalization of X, then $G_n(X) \otimes \mathbb{Q} \subset G_n(X_0)$ and equality holds if X is a simply connected finite CW-complex. $G_*(X_0)$ is called the rational Gottlieb group of X. Useful descriptions of rational Gottlieb groups are given in [3, 14]. However little is known about rational Gottlieb groups of function spaces. In this paper we prove the following result.

Theorem. *Let* $f : X \to S^{2n}$ *be a map such that* $H^{2n}(f, \mathbb{Q}) = \alpha$ *is in the annihilator of* $\tilde{H}^*(X, \mathbb{Q})$. *Then* $\dim G_*(\widetilde{\mathrm{map}}(X, S^{2n}; f)_0) \geq \dim \tilde{H}_*(X, \mathbb{Q})$.

[1]Partial support from the Max-Planck Institute of Mathematics, Bonn, Germany and the African Institute of Mathematical Sciences, Cape Town, South Africa.

2. Rational spherical fibrations

In the sequel we will rely on Sullivan and Quillen theories of rational homotopy, of which details can be found in [13, 11, 14, 4]. We recall here Sullivan and Quillen models of spaces. A Sullivan algebra is a commutative cochain algebra of the form $(\wedge V, d)$, where $V = \oplus_{i\geq 2} V^i$. Moreover $V = \bigcup_{k\geq 0} V(k)$, where $V(0) \subset V(1) \ldots$ such that $d(V(0)) = 0$ and $d(V(k)) \subset \wedge V(k-1)$. It is called minimal if $dV \subset \wedge^{\geq 2} V$. For any commutative differential graded algebra (A, d) of which the cohomology is connected and finite in each degree, there is a unique minimal Sullivan algebra $(\wedge V, d)$ equipped with a quasi-isomorphism $(\wedge V, d) \to (A, d)$. A Sullivan model of a simply connected topological space X is a Sullivan algebra $(\wedge V, d)$ which algebraically models the rational homotopy type of X. It is called minimal if $(\wedge V, d)$ is minimal.

A simply connected space X is called formal if there is a quasi-isomorphism $(\wedge V, d) \to H^*(X, \mathbb{Q})$. Examples of formal spaces include spheres, homogeneous spaces G/H of which $\mathrm{rank}(G) = \mathrm{rank}(H)$ and Kähler manifolds.

A Lie model is a differential graded Lie algebra (L, δ) that determines the rational homotopy type of X. It is a Quillen model if it is of the form $(\mathbb{L}(V), \delta)$ where $\mathbb{L}(V)$ denotes the free graded Lie algebra on the graded vector space $V = \oplus_{i\geq 1} V_i$, and it is called minimal if $\delta V \subset \mathbb{L}^{\geq 2} V$. For a simply connected space X, there exists a minimal Quillen model $(\mathbb{L}(V), \delta)$. Moreover its homology is isomorphic to $\pi_*(\Omega X) \otimes \mathbb{Q}$, the *(rational) homotopy Lie algebra* of X and $V_n \cong H_{n+1}(X, \mathbb{Q})$. A map of spaces $f : X \to Y$ induces a map of Quillen models which we denote abusively by $f : (\mathbb{L}(V), \delta) \to (\mathbb{L}(W), \delta')$. A simply connected space X is called coformal if there is a quasi-isomorphism $(\mathbb{L}(V), \delta) \to \pi_*(\Omega X) \otimes \mathbb{Q}$.

For a simply connected CW-complex, we denote by $aut_1(X)$ the monoid of self homotopy equivalences of X that are homotopic to the identity and $aut_1^\bullet(X)$ its sub monoid of pointed self homotopy equivalences. There is a universal fibration $X \to Baut_1^\bullet(X) \to Baut_1(X)$. It classifies fibrations of which the fibre has the homotopy type of X and the base space is simply connected [2]. Such fibrations can be expressed in terms of Sullivan or Quillen models, hence they are called rational fibrations. Universal rational fibrations have been studied by Sullivan, Schlessinger-Stasheff, Tanré among others [13, 12, 14].

If X has the rational homotopy type of an even dimensional sphere S^{2n}, then the space $Baut_1(X)$ has the rational homotopy type of $K(\mathbb{Q}, 4n)$ [**?**, 14, 5]. Consider the map $\alpha : S^{4n} \to Baut_1(X)$ that corresponds to the generator of $\pi_{4n}(Baut_1(X)) \otimes \mathbb{Q}$. The pullback along the universal fibration gives rise to a rational fibration $S^{2n} \xrightarrow{\imath} E \xrightarrow{p} S^{4n}$. One can describe this fibration by the means of the KS-extension

$$\wedge((x_{4n}, x_{8n-1}), d) \to (\wedge(x_{4n}, x_{8n-1}, x_{2n}, x_{4n-1}), D) \to (\wedge(x_{2n}, x_{4n-1}), d'),$$

where $Dx_{2n} = 0$, $Dx_{4n-1} = x_{2n}^2 - x_{4n}$ (subscripts indicate degrees). The minimal Sullivan model of E is $(\wedge(x_{2n}, x_{8n-1}), d)$, with $dx_{8n-1} = x_{2n}^4$.

As E is formal, it has a Quillen model of the form $(\mathbb{L}(x_{2n-1}, x_{4n-1}, x_{6n-1}), \delta)$ where $\delta x_{2n-1} = 0$, $\delta x_{4n-1} = [x_{2n-1}, x_{2n-1}]$ and $\delta x_{6n-1} = [x_{2n-1}, x_{4n-1}]$. It is the only non trivial deformation of the Quillen minimal model of the product $S^{4n} \times S^{2n}$ in the sense of Tanré [14, chap.7]. Moreover, a non free model of E is given by $(\mathbb{L}(x) \times_\alpha \mathbb{L}(y), d')$, where $\mathbb{L}(x) \times_\alpha \mathbb{L}(y)$ is isomorphic to the product Lie algebra $\mathbb{L}(x) \times \mathbb{L}(y)$ and the differential is defined by $d'x = 0$ and $d'y = [x, x]$.

A Lie model of the fibration $S^{2n} \xrightarrow{\imath} E \xrightarrow{p} S^{4n}$ is then given by

(1)
$$\begin{array}{ccccc} (\mathbb{L}(x), 0) & \xrightarrow{i} & (\mathbb{L}(x) \times_\alpha \mathbb{L}(y), d') & \xrightarrow{p} & (\mathbb{L}(y), 0) \\ & & \simeq \big\uparrow q & & \\ & & (\mathbb{L}(x_{2n-1}, x_{4n-1}, x_{6n-1}), \delta), & & \end{array}$$

where q is the canonical quasi-isomorphism defined by $q(x_{2n-1}) = x$, $q(x_{4n-1}) = y$ and $q(x_{6n-1}) = 0$. As observed in [14], the connecting map of the long homotopy exact sequence of the above fibration is not zero, indeed $\delta : \pi_{4n}(S^{4n}) \otimes \mathbb{Q} \to \pi_{4n-1}(S^{2n}) \otimes \mathbb{Q}$ is an isomorphism.

3. Rational Gottlieb group of $\widetilde{\mathrm{map}}(X, S^{2n})$.

For a map $f : X \to Y$, we denote by $\mathrm{map}(X, Y; f)$ the connected component of $\mathrm{map}(X, Y)$ containing f. Our aim is to compute the rational Gottlieb group of $\widetilde{\mathrm{map}}(X, S^{2n}; f)$, the universal cover of $map(X, S^{2n}; f)$, under additional assumptions on $\tilde{H}^*(f, \mathbb{Q})$. Given the above fibration $S^{2n} \xrightarrow{i} E \xrightarrow{p} S^{4n}$ and a mapping $f : X \to S^{2n}$, one gets a fibration between function spaces

(2)
$$\mathcal{F} \xrightarrow{\iota} \mathrm{map}(X, E; i \circ f) \xrightarrow{\mathrm{map}(p)} \mathrm{map}(X, S^{4n}; c),$$

where c is the constant map.

Lemma 1. *The fibre $\mathcal{F}$ is rationally equivalent to* $\mathrm{map}(X, S^{2n}; f)$.

Proof. Let $f : X \to S^{2n}$. If f is not trivial, we may assume that the induced map $\phi : (\mathbb{L}(V), \delta) \to \mathbb{L}(x)$ verifies either $\phi(v) = x$ or $\phi(v) = [x, x]$. Hence $\mathrm{map}(X, S^{2n})$ has three rational path components.

Let $g \in \mathcal{F}$. Then $p \circ g$ is the constant map, therefore g factors into $\bar{g} : X \to S^{2n}$. We assume first that $\phi(v) = x$. A model of g is given by $\gamma : (\mathbb{L}(V), \delta) \to \mathbb{L}(x) \times_\alpha \mathbb{L}(y), \delta)$ with $\gamma(v) = x$. Therefore a model φ for $\bar{g}$ will verify $\varphi(x) = v$. Hence $\bar{g}$ is rationally equivalent to f.

If $\phi(v) = [x, x]$, then $i \circ f$ is rational trivial and the fibre of $\mathrm{map}(p) : \mathrm{map}(X, E; c) \to \mathrm{map}(X, S^{4n}; c)$ is rationally homotopic to $\mathrm{map}(X, S^{2n}; c)$.

Therefore $\mathrm{map}(X, S^{2n}; f)$ and $\mathrm{map}(X, S^{2n}; c)$ have same rational homotopy type [10].

Finally if $\phi(v) = 0$, then both f and $\bar{g}$ are trivial and the fibre has the rational homotopy type of $\mathrm{map}(X, S^{2n}, c)$. In all three cases, the fibre $\mathcal{F}$ has the rational homotopy type of $\mathrm{map}(X, S^{2n}; f)$. $\square$

Let $(\mathbb{L}(V), \delta)$ be the Quillen minimal model of X and $(T(V), d)$ its enveloping algebra.

It is known that there is an acyclic differential $T(V)$-module of the form $(T(V) \otimes (\mathbb{Q} \oplus sV), D)$ [1, 9]. The differential D is given by

$$D(v \otimes 1) = dv \otimes 1, \quad Dsv = v \otimes 1 - S(dv \otimes 1),$$

where S is the $\mathbb{Q}$-graded vector space map (of degree 1) defined by

$$S(v \otimes 1) = 1 \otimes sv, \quad S(1 \otimes (\mathbb{Q} \oplus sV)) = 0,$$

$$S(ax \otimes 1) = (-1)^{|a|} aS(x \otimes 1), \ \forall a \in TV, \ |x| > 0.$$

Recall that if A is a differential graded algebra and (M, d_M) and (N, d_N) are differential graded A-modules, $\mathrm{Hom}_A(M, N)$ is a differential vector space of which the differential is defined by $Df = d_N \circ f - (-1)^{|f|} f \circ d_M$.

Let $f : X \to Y$ be a map between spaces, $\phi : (\mathbb{L}(V), \delta) \to (\mathbb{L}(W), \delta')$ its Quillen model and $U\phi : TV \to TW$ the induced mapping between enveloping algebras. The adjoint action of TW on $\mathbb{L}(W)$ combined with $U\phi$ induces a $T(V)$-module structure on $\mathbb{L}(W)$. For $n \geq 1$, it was shown that there is an isomorphism [6]

$$\begin{aligned} \pi_n(\Omega\, \mathrm{map}(X, Y; f)) \otimes \mathbb{Q} &\cong \mathrm{Ext}^n_{TV}(\mathbb{Q}, \mathbb{L}(W)) \\ &\cong H_n((\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W)), D). \end{aligned}$$

Let $P = TV \otimes (\mathbb{Q} \oplus sV)$. The complex $C = \mathrm{Hom}_{TV}(P, \mathbb{L}(W))$ is not positively graded. If we put

$$\tilde{C}_i = \begin{cases} C_i & \text{for } i > 1 \\ Z(C)_1 & \text{for } i = 1, \end{cases}$$

then the complex $\tilde{C}$ computes the rational homotopy groups of the universal cover $\widetilde{\mathrm{map}}(X, Y; f)$ of $\mathrm{map}(X, Y; f)$.

The long exact sequence of the fibration

$$\mathrm{map}(X, S^{2n}; f) \to \mathrm{map}(X, E; i \circ f) \to \mathrm{map}(X, S^{4n}; c)$$

can be computed from the exact sequence

(3)

$$\mathrm{Hom}(\mathbb{Q} \oplus sV, \mathbb{L}(x)) \xrightarrow{\imath} \mathrm{Hom}(\mathbb{Q} \oplus sV, \mathbb{L}(x) \times_\alpha \mathbb{L}(y)) \xrightarrow{\psi} \mathrm{Hom}(\mathbb{Q} \oplus sV, \mathbb{L}(y)).$$

As the differential on $\mathrm{Hom}(\mathbb{Q} \oplus sV, \mathbb{L}(y))$ is zero, then

$$\pi_*(\widetilde{\mathrm{map}}(X, S^{4n}; c) \otimes \mathbb{Q} \cong \mathrm{Hom}(\mathbb{Q} \oplus sV, \mathbb{L}(y)).$$

The long exact sequence of the fibration (2) becomes

(4)

$$\to \mathrm{Ext}_{TV}(\mathbb{Q}, \mathbb{L}(x)\times_\alpha \mathbb{L}(y)) \overset{H(\psi)}{\to} \mathrm{Hom}(\mathbb{Q}\oplus sV, \mathbb{L}(y)) \overset{\delta}{\to} \mathrm{Ext}_{TV}(\mathbb{Q}, \mathbb{L}(x)) \to$$

We have the following result.

Theorem 2. *Let $f : X \to S^{2n}$ be a map such that $H^{2n}(f,\mathbb{Q}) = \alpha$ is in the annihilator of $\tilde{H}^*(X,\mathbb{Q})$. Then* $\dim G_*(\widetilde{\mathrm{map}}(X, S^{2n}; f)_0) \geq \dim \tilde{H}_*(X,\mathbb{Q})$.

Proof. We use the connecting map of the long exact sequence (4).

Take $\theta \in \mathrm{Hom}(\mathbb{Q}\oplus sV, \mathbb{L}(y))$. We need to compute $\delta(\theta) \in \mathrm{Ext}_{TV}(\mathbb{Q}, \mathbb{L}(x))$.

We first consider the case where $\theta(1) = y$ and $\theta(sV) = 0$. There is $\bar\theta \in \mathrm{Hom}(\mathbb{Q}\oplus sV, \mathbb{L}(x)\times_\alpha \mathbb{L}(y))$ such that $\psi(\bar\theta) = \theta$. One can choose $\bar\theta(1) = y$ and zero on sV. As $(D\bar\theta)(1) = [x,x]$, hence $\delta\theta \in Ext_{TV}(\mathbb{Q}\oplus sV, \mathbb{L}(x))$ is represented by a cocycle $\gamma \in \mathrm{Hom}(\mathbb{Q}\oplus sV, \mathbb{L}(x))$ such that $\gamma(1) = [x,x]$ and $\gamma(sV) = 0$. This cocycle cannot be a coboundary as $[x,x]$ is a non zero homology class in $\mathbb{L}(x)$. Consequently $\delta(\theta) \in \mathrm{Ext}_{TV}(\mathbb{Q}, \mathbb{L}(x))$ is not zero.

Let $\{v_1, v_2, \dots\}$ be a basis of V. Consider now $\theta \in \mathrm{Hom}(\mathbb{Q}\oplus sV, \mathbb{L}(y))$ such that $\theta(sv_k) = y$ and zero otherwise. In the same way, we define $\bar\theta \in \mathrm{Hom}(\mathbb{Q}\oplus sV, \mathbb{L}(x)\times_\alpha \mathbb{L}(y))$ by $\bar\theta(sv_k) = y$ and zero otherwise. For any $w \in \{v_1, v_2, \dots\}$,

$$\begin{aligned}(D\bar\theta)(sw) &= d'\bar\theta(sw) - (-1)^{|\theta|}\bar\theta(w\otimes 1 - S(dw\otimes 1))\\ &= d'\bar\theta(sw) - (-1)^{|\theta||sw|}[\phi(w), \bar\theta(1)] + (-1)^{|\theta|}\bar\theta(S(dw\otimes 1))\\ &= d'\bar\theta(sw) + (-1)^{|\theta|}\bar\theta(S(dw\otimes 1)) \quad (\text{as} \quad \bar\theta(1) = 0).\end{aligned}$$

As dv_k is a polynomial in variables of degree less that $|v_k|$, we deduce that $\bar\theta(S(dv_k\otimes 1)) = 0$, therefore $(D\bar\theta)(sv_k) = [x,x]$. Moreover for $v_j \neq v_k$, $(D\bar\theta)(sv_j) = -(-1)^{|\theta||sv_j|}\bar\theta(S(dv_j\otimes 1))$. Therefore we define $\tilde\theta \in \mathrm{Hom}(\mathbb{Q}\oplus sV, \mathbb{L}(x))$ by $\tilde\theta(1) = 0$, $\tilde\theta(sv_k) = [x,x]$ and $\tilde\theta(sv_j) = -(-1)^{|\theta||sv_j|}\bar\theta(S(dv_j\otimes 1))$ for $v_j \neq v_k$.

We now show that $\tilde\theta$ cannot be a coboundary. On the contrary assume that there exists $\varphi \in \mathrm{Hom}(\mathbb{Q}\oplus sV, \mathbb{L}(x))$ such that $D\varphi = \tilde\theta$. Let $v \in V$ be the dual of $\alpha \in H^{2n}(X,\mathbb{Q})$. We suppose that $v_k \neq v$ so that $\phi(v_k) = 0$ or $\phi(v_k) = r[x,x]$.

Therefore

$$\begin{aligned}(D\varphi)(sv_k) &= \delta\varphi(sv_k) - (-1)^{|\varphi|}\varphi(v_k\otimes 1 - S(dv_k\otimes 1))\\ &= -(-1)^{|\varphi||sv_k|}[\phi(v_k), \varphi(1)] + (-1)^{|\varphi|}\varphi(S(dv_k\otimes 1))\\ &= (-1)^{|\varphi||sv_k|}\varphi(S(dv_k\otimes 1)).\end{aligned}$$

We write $dv_k = d_2v_k + d_3v_k + \dots$, where $d_iv_k \in T^i(V)$. Therefore $S(dv_k\otimes 1) = S(d_2v_k\otimes 1) + S(d_3v_k\otimes 1) + \dots$. Moreover $\varphi(S(d_iv_k\otimes 1)) \in \mathbb{L}^{\geq 3}(x)$ for $i \geq 3$. But the polynomial d_2v_k does not contain v, as the dual of v annihilates $\tilde{H}^*(X,\mathbb{Q})$. Therefore $\varphi(S(d_2v_k\otimes 1) = 0$ as well. We conclude that each element in $\{1, v_1, v_2, \dots\} \cong H^*(X,\mathbb{Q})$, distinct from v, gives rise to a non zero Gottlieb element $\tilde{\theta}_k$ in $\pi_*(\Omega\widetilde{\mathrm{map}}(X, S^{2n}; f)) \otimes \mathbb{Q} \cong \mathrm{Ext}_{TV}(\mathbb{Q}, \mathbb{L}(x))$. By construction the $\{\tilde{\theta}_k\}$ are linearly independent. □

From the above theorem, we can immediately deduce the following result.

Corollary 3. *If $f : S^{2n} \vee X \to S^{2n}$ is the map that consists in collapsing X onto a point, then the rational Gottlieb group of $\widetilde{\mathrm{map}}(S^{2n} \vee X, S^{2n}; f) \geq \dim H^*(X, \mathbb{Q})$.*

Example 4. *We consider the map $f : \mathbb{C}P(2) \to S^4$ of which the minimal model is given by*

$$\phi : (\wedge(x_4, x_7), d) \to (\wedge(x_2, x_5), d),$$

where $dx_4 = 0$, $dx_7 = x_4^2$, $dx_2 = 0$, $dx_5 = x_2^3$ and $\phi(x_4) = x_2^2$, $\phi(x_7) = x_2x_5$. As the image of $\tilde{H}^*(f, \mathbb{Q})$ annihilates $\tilde{H}^*(\mathbb{C}P(2), \mathbb{Q})$, we deduce that the rational Gottlieb group of $\widetilde{\mathrm{map}}(\mathbb{C}P(2), S^4; f)$ is of dimension at least 2.

References

[1] J.F. Adams and P.J. Hilton, *On the chain algebra of a loop space*, Comment. Math. Helv. **30** (1956), 305–330.

[2] E. Dror and A. Zabrodsky, *Unipotency and nilpotency in homotopy equivalences*, Topology **18** (1979), no. 4, 187–197.

[3] Y. Félix and S. Halperin, *Rational LS category and its applications*, Trans. Amer. Math. Soc. **273** (1982), 1–38.

[4] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, no. 205, Springer-Verlag, New-York, 2001.

[5] J.-B. Gatsinzi, *LS-category of classifying spaces*, Bull. Belg. Math. Soc. **2** (1995), 121–126.

[6] J.-B. Gatsinzi and R. Kwashira, *Rational homotopy groups of function spaces*, Homotopy Theory of Function Spaces and Related Topics (Y. Félix, G. Lupton, and S. Smith, eds.), Contemporary Mathematics, vol. 519, American Mathematical Society, Providence, 2010, p. 105-114.

[7] D.H. Gottlieb, *Evaluation subgroups of homotopy groups*, Amer. J. of Math. **91** (1969), 729–756.

[8] D.H. Gottlieb, *The total space of universal fibrations*, Pacific J. Math. **46** (1973), 415–417.

[9] J.-M. Lemaire, *Algèbres connexes et homologie des espaces de lacets*, Lecture Notes in Mathematics, no. 422, Springer, Berlin, 1974.

[10] J.M. Møller and M. Raussen, *Rational homotopy of spaces of maps into spheres and complex projective spaces*, Trans. Amer. Math. Soc **292** (1985), 721–732.

[11] D. Quillen, *Rational homotopy theory*, Annals of Math. **90** (1969), 205-295.

[12] M. Schlessinger and J. Stasheff, *Deformations theory and rational homotopy type*, preprint, 1982.

[13] D. Sullivan, *Infinitesimal computations in topology*, Publ. I.H.E.S. **47** (1977), 269–331.

[14] D. Tanré, *Homotopie Rationnelle: Modèles de Chern, Quillen, Sullivan*, Lecture Notes in Mathematics, no. 1025, Springer, Berlin, 1983.

Received: November, 2011