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Semigroups of Linear Transformations with Invariant Subspaces

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Abstract

Let V be a vector space and let $T(V)$ denote the semigroup (under composition) of all linear transformations from V into itself. For a fixed subspace W of V, let $S(V, W)$ be the subsemigroup of $T(V)$, consisting of all linear transformations on V which leave a subspace W of V invariant. The purpose of this paper is to characterize Green's relations and ideals on $S(V, W)$ and prove that $S(V, W)$ is never isomorphic to $T(U)$ for any vector space U when W is a non-zero proper subspace of V.

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1 Introduction

Let $T(X)$ denote the semigroup (under composition) of all transformations from X into itself. For a fixed nonempty subset Y of X, let

$$S(X, Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}.$$

Then $S(X, Y)$ is a semigroup of total transformations of X which leave a subset Y of X invariant. Magill [4] introduced and studied the semigroup $S(X, Y)$

in 1966. Later in 2006 Nenthein and Kemprasit [5] characterized its regular elements. Recently, we described Green's relations and ideals on $S(X,Y)$ (see [2] for details).

For a vector space V, let $T(V)$ be the semigroup (under composition) of all linear transformations from V into itself. It is known that $T(V)$ is a regular semigroup [1, Volume 1, exercise 2.2.6]. Here, we consider the subsemigroup of $T(V)$, analogous to $S(X,Y)$, defined by

$$S(V,W) = \{\alpha \in T(V) : W\alpha \subseteq W\},$$

where W is a subspace of V and $W\alpha$ denotes the range of W under α. Note that id_V and 0, the identity map and the zero map on V respectively, belong to $S(V,W)$. In addition, $S(V,W)$ contains

$$L(V,W) = \{\alpha \in T(V) : V\alpha \subseteq W\}$$

as an ideal of $S(V,W)$. If $\{0\} \neq W \subsetneq V$, then $S(V,W)$ is not a 0-simple semigroup since $id_V \in S(V,W)\setminus L(V,W)$ [5, page 9].

In 2006, Nenthein and Kemprasit [5] defined the semigroup $S(V,W)$ and characterized the regular elements of $S(V,W)$. The authors proved that for $\alpha \in S(V,W)$, α is a regular element of $S(V,W)$ if and only if $V\alpha \cap W = W\alpha$, and showed that, $S(V,W)$ is regular if and only if either $W = V$ or $W = \{0\}$. Here, in Section 2, we prove that $S(V,W)$ is never isomorphic to $T(U)$ for any vector space U when W is a non-zero proper subspace of V. We also show that $S(V,W)$ is almost never isomorphic to $S(X,Y)$. In Section 3, we characterize Green's relations on $S(V,W)$, and we describe its ideals in Section 4.

In this paper, the vector space V we consider, $\dim V$ can be finite or infinite. The cardinality of a set A is denoted by $|A|$. Also $X = A\dot{\cup}B$ means X is a disjoint union of A and B.

2 Isomorphism of $S(V,W)$

For a vector space V, let $T(V)$ denote the semigroup (under composition) of all linear transformations from V into itself. For a fixed subspace W of V, let

$$S(V,W) = \{\alpha \in T(V) : W\alpha \subseteq W\},$$

where $W\alpha$ denotes the range of W under α. Then $S(V,W)$ is a subsemigroup of $T(V)$, which contains id_V and 0, the identity map and the zero map on V respectively. If $W = \{0\}$ or $W = V$, we have $S(V,W) = T(V)$ which is a regular semigroup.

In general $S(V,W)$ is not a regular subsemigroup of $T(V)$. The following lemma characterizes the regular elements of $S(V,W)$, and gives a necessary and sufficient condition for $S(V,W)$ to be regular.

Lemma 1. ([5]) *The following statements hold for the semigroup* $S(V, W)$.

(1) *For* $\alpha \in S(V, W)$, α *is a regular element of* $S(V, W)$ *if and only if* $V\alpha \cap W = W\alpha$.

(2) $S(V, W)$ *is regular if and only if either* $W = V$ *or* $W = \{0\}$.

From [2, Theorem 1], we see that $S(X, Y)$ is isomorphic to $T(Z)$ if and only if $X = Y$ and $|Y| = |Z|$. Here, the following corollary shows that $S(V, W)$ is almost never isomorphic to $T(U)$ for any vector space U.

Corollary 1. *If* $\{0\} \neq W \subsetneq V$, *then* $S(V, W)$ *is not isomorphic to* $T(U)$ *for any vector space* U.

Proof. Suppose that $\{0\} \neq W \subsetneq V$. By Lemma 1, we have $S(V, W)$ is not regular, thus $S(V, W) \ncong T(U)$. □

For convenience, we adopt the convention introduced in Clifford and Preston [1, Volume 2, page 241]: namely, if $\alpha \in T(X)$ then we write

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$, and that $X\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i$.

Similarly, we can use this notation for elements in $T(V)$. To construct a map $\alpha \in T(V)$, we first choose a basis $\{e_i\}$ for V and a subset $\{a_i\}$ of V, and then let $e_i\alpha = a_i$ for each $i \in I$ and extend this map linearly to V. To shorten this process, we simply say, given $\{e_i\}$ and $\{a_i\}$ within the context, then for each $\alpha \in T(V)$, we can write

$$\alpha = \begin{pmatrix} e_i \\ a_i \end{pmatrix}.$$

A subspace U of V generated by a linearly independent subset $\{e_i\}$ of V is denoted by $\langle e_i \rangle$ and when we write $U = \langle e_i \rangle$, we mean that the set $\{e_i\}$ is a basis of U, and we have $\dim U = |I|$. For each $\alpha \in T(V)$, the kernel and the range of α denoted by $\ker \alpha$ and $V\alpha$, respectively.

We note that $S(V, W)$ always contains a zero element, the zero map. But, for $S(X, Y)$ we have the following lemma.

Lemma 2. $S(X, Y)$ *has a zero element if and only if* $|Y| = 1$.

Proof. Assume that $|Y| = 1$. Let $Y = \{a\}$ and define

$$\alpha = \begin{pmatrix} X \\ a \end{pmatrix} \in S(X, Y).$$

Then $\alpha\beta = \alpha = \beta\alpha$ for all $\beta \in S(X, Y)$ and therefore α is a zero element of $S(X, Y)$. Conversely, assume that $S(X, Y)$ has a zero element, say α. Suppose that $|Y| > 1$. Let $b, c \in Y$ be such that $b \neq c$ and define

$$\beta = \begin{pmatrix} X \\ b \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} X \\ c \end{pmatrix}.$$

Then $\beta, \gamma \in S(X, Y)$ and $\beta \neq \gamma$. Since α is a zero element, we have $\alpha\beta = \alpha = \alpha\gamma$. But $\beta = \alpha\beta$ and $\gamma = \alpha\gamma$. Thus $\beta = \alpha\beta = \alpha = \alpha\gamma = \gamma$ which is a contradiction. Therefore, $|Y| = 1$. □

Thus, if $|Y| > 1$ then $S(X, Y)$ has no zero elements. So we have the following corollary.

Corollary 2. *Let Y be a fixed nonempty subset of X. If $|Y| > 1$, then $S(X, Y)$ is not isomorphic to $S(V, W)$ for any subspace W of a vector space V.*

3 Green's relations on $S(V, W)$

It is well-known that $\alpha\mathcal{L}\beta$ in $T(V)$ if and only if $V\alpha = V\beta$; and $\alpha\mathcal{R}\beta$ in $T(V)$ if and only if $\ker\alpha = \ker\beta$ (see [1, Volume 1, exercise 2.2.6]). But, for $S(V, W)$ we have the following lemmas.

Lemma 3. *Let $\alpha, \beta \in S(V, W)$. Then $\alpha = \gamma\beta$ for some $\gamma \in S(V, W)$ if and only if $V\alpha \subseteq V\beta$ and $W\alpha \subseteq W\beta$. Consequently, $\alpha\mathcal{L}\beta$ if and only if $V\alpha = V\beta$ and $W\alpha = W\beta$.*

Proof. Assume that $\alpha = \gamma\beta$ for some $\gamma \in S(V, W)$. Then $V\alpha = V\gamma\beta = (V\gamma)\beta \subseteq V\beta$ and $W\alpha = W\gamma\beta = (W\gamma)\beta \subseteq W\beta$ since $\gamma \in S(V, W)$.

Conversely, suppose that $V\alpha \subseteq V\beta$ and $W\alpha \subseteq W\beta$. Then $W(\alpha|_W) \subseteq W(\beta|_W)$ where $\alpha|_W, \beta|_W \in T(W)$. Hence $\alpha|_W = \delta(\beta|_W)$ for some $\delta \in T(W)$. Write $W = \langle w_i \rangle$ and $V = \langle w_i \rangle \oplus \langle v_j \rangle$. So $w_i\alpha = w_i\delta\beta$. Now, for each v_j, there exists some $v'_j \in V$ such that $v_j\alpha = v'_j\beta$ since $V\alpha \subseteq V\beta$. Thus we extend $\delta \in T(W)$ to $\gamma \in T(V)$ by

$$\gamma = \begin{pmatrix} w_i & v_j \\ w_i\delta & v'_j \end{pmatrix}.$$

Then $\gamma \in S(V, W)$ and $\alpha = \gamma\beta$ as required. □

Lemma 4. *Let $\alpha, \beta \in S(V, W)$. Then $\alpha = \beta\gamma$ for some $\gamma \in S(V, W)$ if and only if* $\ker\beta \subseteq \ker\alpha$ *and* $W\beta^{-1} \subseteq W\alpha^{-1}$. *Consequently, $\alpha\mathcal{R}\beta$ if and only if* $\ker\alpha = \ker\beta$ *and* $W\alpha^{-1} = W\beta^{-1}$.

Proof. Assume that $\alpha = \beta\gamma$ for some $\gamma \in S(V, W)$. If $a \in \ker\beta$, then $a\alpha = a\beta\gamma = (a\beta)\gamma = 0\gamma = 0$ and hence $\ker\beta \subseteq \ker\alpha$. If $b \in W\beta^{-1}$, then $b\beta \in W$ and so $b\alpha = b\beta\gamma = (b\beta)\gamma \in W\gamma \subseteq W$. Thus $b \in W\alpha^{-1}$ implies $W\beta^{-1} \subseteq W\alpha^{-1}$.

To prove the converse, we assume that $\ker\beta \subseteq \ker\alpha$ and $W\beta^{-1} \subseteq W\alpha^{-1}$. Let $V\beta = \langle v_i\rangle$. Thus $V = \langle v_i\rangle \oplus \langle u_j\rangle$. Since $v_i \in V\beta$, there exists $z_i \in V$ such that $v_i = z_i\beta$. So we define $\gamma \in T(V)$ by

$$\gamma = \begin{pmatrix} v_i & u_j \\ z_i\alpha & 0 \end{pmatrix}.$$

Then γ is well-defined since $\ker\beta \subseteq \ker\alpha$. Now, we prove that $\gamma \in S(V, W)$. Since W is a subspace of V, we can write $W = \langle w_k\rangle$ where $\{w_k\} \subseteq \{v_i, u_j\}$. If $w_k = u_j$, then $w_k\gamma = u_j\gamma = 0 \in W$. If $w_k = v_i$, then there exists $z_i \in V$ such that $w_k = v_i = z_i\beta$. Thus $z_i \in w_k\beta^{-1} \subseteq W\beta^{-1} \subseteq W\alpha^{-1}$. So $w_k\gamma = z_i\alpha \in W$. Also, we have $v\beta\gamma = (v\beta)\gamma = v\alpha$ for all $v \in V$ by the definition of γ. □

Theorem 1. *Let $\alpha, \beta \in S(V, W)$. Then $\alpha\mathcal{D}\beta$ if and only if* $\dim(W\alpha) = \dim(W\beta)$, $\dim((V\alpha \cap W)/W\alpha) = \dim((V\beta \cap W)/W\beta)$ *and* $\dim(V\alpha/(V\alpha \cap W)) = \dim(V\beta/(V\beta \cap W))$.

Proof. We first note that for any $\alpha \in S(V, W)$, $\dim(W\alpha) = \dim(W/(W \cap \ker\alpha))$, $\dim((V\alpha \cap W)/W\alpha) = \dim(W\alpha^{-1}/(W\alpha)\alpha^{-1})$ and $\dim(V\alpha/(V\alpha \cap W)) = \dim(V/W\alpha^{-1})$. Assume that $\alpha\mathcal{D}\beta$. Then $\alpha\mathcal{L}\gamma\mathcal{R}\beta$ for some $\gamma \in S(V, W)$. So $V\alpha = V\gamma$ and $W\alpha = W\gamma$ by Lemma 3, and $\ker\gamma = \ker\beta$ and $W\gamma^{-1} = W\beta^{-1}$ by Lemma 4. Thus $\dim(W\alpha) = \dim(W\gamma) = \dim(W/(W \cap \ker\gamma)) = \dim(W/(W \cap \ker\beta)) = \dim(W\beta)$, and $\dim(V\alpha/(V\alpha \cap W)) = \dim(V\gamma/(V\gamma \cap W)) = \dim(V/W\gamma^{-1}) = \dim(V/W\beta^{-1}) = \dim(V\beta/(V\beta \cap W))$. From $\ker\gamma = \ker\beta$, we have $(W\gamma)\gamma^{-1} = (W\beta)\beta^{-1}$. So $\dim((V\alpha \cap W)/W\alpha) = \dim((V\gamma \cap W)/W\gamma) = \dim(W\gamma^{-1}/(W\gamma)\gamma^{-1}) = \dim(W\beta^{-1}/(W\beta)\beta^{-1}) = \dim((V\beta \cap W)/W\beta)$.

Conversely suppose that the conditions hold. Assume that $\dim(W\alpha) = |I| = \dim(W\beta)$, $\dim((V\alpha \cap W)/W\alpha) = |J| = \dim((V\beta \cap W)/W\beta)$ and $\dim(V\alpha/(V\alpha \cap W)) = |K| = \dim(V\beta/(V\beta \cap W))$. Then we can write $W\alpha = \langle w_i\alpha\rangle$ and $W\beta = \langle w_i'\beta\rangle$; $(V\alpha \cap W)/W\alpha = \langle u_j\alpha + W\alpha\rangle$ and $(V\beta \cap W)/W\beta = \langle u_j'\beta + W\beta\rangle$; $V\alpha/(V\alpha \cap W) = \langle v_k\alpha + V\alpha \cap W\rangle$ and $V\beta/(V\beta \cap W) = \langle v_k'\beta + V\beta \cap W\rangle$. Since $W\alpha = \langle w_i\alpha\rangle$ and $(V\alpha \cap W)/W\alpha = \langle u_j\alpha + W\alpha\rangle$, we have $V\alpha \cap W = \langle w_i\alpha\rangle \oplus \langle u_j\alpha\rangle$. Thus $V\alpha = \langle w_i\alpha\rangle \oplus \langle u_j\alpha\rangle \oplus \langle v_k\alpha\rangle$ (since $V\alpha/(V\alpha \cap W) = \langle v_k\alpha + V\alpha \cap W\rangle$). Similarly, we get $V\beta = \langle w_i'\beta\rangle \oplus \langle u_j'\beta\rangle \oplus \langle v_k'\beta\rangle$.

Let $\ker\alpha = \langle z_r\rangle$ and $\ker\beta = \langle z_t\rangle$. So $V = \langle z_r\rangle \oplus \langle w_i\rangle \oplus \langle u_j\rangle \oplus \langle v_k\rangle$ and $V = \langle z_t\rangle \oplus \langle w_i'\rangle \oplus \langle u_j'\rangle \oplus \langle v_k'\rangle$. Thus we can write

$$\alpha = \begin{pmatrix} z_r & w_i & u_j & v_k \\ 0 & w_i\alpha & u_j\alpha & v_k\alpha \end{pmatrix} \text{ and } \beta = \begin{pmatrix} z_t & w_i' & u_j' & v_k' \\ 0 & w_i'\beta & u_j'\beta & v_k'\beta \end{pmatrix}.$$

Define

$$\mu = \begin{pmatrix} z_t & w_i' & u_j' & v_k' \\ 0 & w_i\alpha & u_j\alpha & v_k\alpha \end{pmatrix}.$$

Then $W\mu = \langle w_i\alpha\rangle \subseteq W$ and hence $\mu \in S(V,W)$. Also, we have $W\mu = \langle w_i\alpha\rangle = W\alpha$ and $V\mu = \langle w_i\alpha\rangle \oplus \langle u_j\alpha\rangle \oplus \langle v_k\alpha\rangle = V\alpha$. So $\alpha\mathcal{L}\mu$ by Lemma 3. Also, we have $\ker\mu = \langle z_t\rangle = \ker\beta$ and $W\mu^{-1} = \langle z_t\rangle \oplus \langle w_i'\rangle \oplus \langle u_j'\rangle = W\beta^{-1}$, thus $\mu\mathcal{R}\beta$ by Lemma 4 and therefore $\alpha\mathcal{D}\beta$. □

Theorem 2. *Let $\alpha, \beta \in S(V,W)$. Then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in S(V,W)$ if and only if $\dim(V\alpha) \le \dim(V\beta), \dim(W\alpha) \le \dim(W\beta)$ and $\dim(V\alpha/(V\alpha \cap W)) \le \dim(V\beta/(V\beta \cap W))$. Consequently, $\alpha\mathcal{J}\beta$ if and only if $\dim(V\alpha) = \dim(V\beta), \dim(W\alpha) = \dim(W\beta)$ and $\dim(V\alpha/(V\alpha \cap W)) = \dim(V\beta/(V\beta \cap W))$.*

Proof. Assume that $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in S(V,W)$. Then $V\alpha = V\lambda\beta\mu = (V\lambda)\beta\mu \subseteq V\beta\mu$ and $W\alpha = W\lambda\beta\mu = (W\lambda)\beta\mu \subseteq W\beta\mu$. Since $\dim(V\beta\mu) \le \dim(V\beta)$ and $\dim(W\beta\mu) \le \dim(W\beta)$, so we have $\dim(V\alpha) \le \dim(V\beta)$ and $\dim(W\alpha) \le \dim(W\beta)$. Also,

$$\begin{aligned} \dim(V\alpha/(V\alpha \cap W)) &= \dim(V\lambda\beta\mu/(V\lambda\beta\mu \cap W)) \\ &= \dim(V(\lambda)\beta\mu/(V(\lambda)\beta\mu \cap W)) \\ &\le \dim(V\beta\mu/(V\beta\mu \cap W)) \\ &\le \dim(V\beta/(V\beta \cap W)), \end{aligned}$$

since there exists a map from $V\beta/(V\beta \cap W)$ onto $V\beta\mu/(V\beta\mu \cap W)$.

Conversely, assume that the conditions hold. From $\dim(W\alpha) \le \dim(W\beta)$, we can write $W\alpha = \langle w_i\alpha\rangle$ and $W\beta = \langle w_i'\beta\rangle \oplus \langle w_l'\beta\rangle$. Since $W\alpha$ is a subspace of $V\alpha \cap W$ and $W\beta$ is a subspace of $V\beta \cap W$, write $V\alpha \cap W = \langle w_i\alpha\rangle \oplus \langle u_j\alpha\rangle$ and $V\beta \cap W = \langle w_i'\beta\rangle \oplus \langle w_l'\beta\rangle \oplus \langle u_m'\beta\rangle$. And from $\dim(V\alpha/(V\alpha \cap W)) \le \dim(V\beta/(V\beta \cap W))$, we can write $V\alpha/(V\alpha \cap W) = \langle v_k\alpha + V\alpha \cap W\rangle$ and $V\beta/(V\beta \cap W) = \langle v_k'\alpha + V\beta \cap W\rangle \oplus \langle v_n'\beta + V\beta \cap W\rangle$. Then $V\alpha = \langle w_i\alpha\rangle \oplus \langle u_j\alpha\rangle \oplus \langle v_k\alpha\rangle$ and $V\beta = \langle w_i'\beta\rangle \oplus \langle w_l'\beta\rangle \oplus \langle u_m'\beta\rangle \oplus \langle v_n'\beta\rangle \oplus \langle v_k'\beta\rangle$. Let $\ker\alpha = \langle z_r\rangle$ and $\ker\beta = \langle z_t\rangle$. So $V = \langle z_r\rangle \oplus \langle w_i\rangle \oplus \langle u_j\rangle \oplus \langle v_k\rangle$ and $V = \langle z_t\rangle \oplus \langle w_i'\rangle \oplus \langle w_l'\rangle \oplus \langle u_m'\rangle \oplus \langle v_n'\rangle \oplus \langle v_k'\rangle$. Then

$$\alpha = \begin{pmatrix} z_r & w_i & u_j & v_k \\ 0 & w_i\alpha & u_j\alpha & v_k\alpha \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} z_t & w'_i & w'_l & u'_m & v'_n & v'_k \\ 0 & w'_i\beta & w'_l\beta & u'_m\beta & v'_n\beta & v'_k\beta \end{pmatrix}.$$

Since $\dim(V\alpha) \leq \dim(V\beta)$, we have $|I|+|J|+|K| \leq |I|+|L|+|M|+|N|+|K|$. To find $\lambda, \mu \in S(V,W)$ such that $\alpha = \lambda\beta\mu$, we follow the ideas of [2, Theorem 5]. Now, we consider two cases:

Case 1 : $|J| \leq |L|+|M|+|N|$. Let $L \cup M \cup N = P\dot{\cup}Q$ where $|P| = |J|$. Then we can write $\{w'_l\} \cup \{u'_m\} \cup \{v'_n\} = \{s_p\} \cup \{s_q\}$ and rewrite β as follow:

$$\beta = \begin{pmatrix} z_t & w'_i & s_p & s_q & v'_k \\ 0 & w'_i\beta & s_p\beta & s_q\beta & v'_k\beta \end{pmatrix}.$$

Since $|J| = |P|$, there is a bijection $\varphi : J \to P$. Now define

$$\lambda = \begin{pmatrix} z_r & w_i & u_j & v_k \\ 0 & w'_i & s_{j\varphi} & v'_k \end{pmatrix}.$$

Then $W\lambda = \langle w'_i \rangle \subseteq W$ and so $\lambda \in S(V,W)$. From $V\beta = \langle w'_i\beta \rangle \oplus \langle s_p\beta \rangle \oplus \langle s_q\beta \rangle \oplus \langle v'_k\beta \rangle$, we let $V = \langle w'_i\beta \rangle \oplus \langle s_p\beta \rangle \oplus \langle s_q\beta \rangle \oplus \langle v'_k\beta \rangle \oplus \langle z'_s \rangle$ and define

$$\mu = \begin{pmatrix} w'_i\beta & s_{j\varphi}\beta & v'_k\beta & \{s_q\beta, z'_s\} \\ w_i\alpha & u_j\alpha & v_k\alpha & 0 \end{pmatrix}.$$

So $\mu \in S(V,W)$ and $\alpha = \lambda\beta\mu$.

Case 2 : $|J| > |L|+|M|+|N|$. Then $\dim(V\beta)$ is infinite (for if $\dim(V\beta)$ is finite, then $\dim(V\alpha) = |I|+|J|+|K| > |I|+|L|+|M|+|N|+|K| = \dim(V\beta)$ which is a contradiction). Hence $|J| \leq |I|$ or $|J| \leq |K|$ are infinite cardinals. If $|J| \leq |I|$ is an infinite cardinal, then write $I = P\dot{\cup}Q$ where $|P| = |I|, |Q| = |J|$. Thus we can write $\{w'_i\} = \{s_p\} \cup \{s_q\}$ and rewrite β as follow:

$$\beta = \begin{pmatrix} z_t & s_p & s_q & w'_l & u'_m & v'_n & v'_k \\ 0 & s_p\beta & s_q\beta & w'_l\beta & u'_m\beta & v'_n\beta & v'_k\beta \end{pmatrix}.$$

Since $|I| = |P|$ and $|J| = |Q|$, there are bijections $\varphi : I \to P$ and $\psi : J \to Q$. And from $V\beta = \langle s_p\beta \rangle \oplus \langle s_q\beta \rangle \oplus \langle w'_l\beta \rangle \oplus \langle u'_m\beta \rangle \oplus \langle v'_n\beta \rangle \oplus \langle v'_k\beta \rangle$, we can write $V = \langle s_p\beta \rangle \oplus \langle s_q\beta \rangle \oplus \langle w'_l\beta \rangle \oplus \langle u'_m\beta \rangle \oplus \langle v'_n\beta \rangle \oplus \langle v'_k\beta \rangle \oplus \langle z'_t \rangle$. Then define λ and μ as follows:

$$\lambda = \begin{pmatrix} z_r & w_i & u_j & v_k \\ 0 & s_{i\varphi} & s_{j\psi} & v'_k \end{pmatrix},$$

and

$$\mu = \begin{pmatrix} s_{i\varphi}\beta & s_{j\psi}\beta & v'_k\beta & \{w'_l\beta, u'_m\beta, v'_n\beta, z'_t\} \\ w_i\alpha & u_j\alpha & v_k\alpha & 0 \end{pmatrix}.$$

So, we see that $\lambda, \mu \in S(V,W)$ and $\alpha = \lambda\beta\mu$. For the case $|J| \leq |K|$ is an infinite cardinal, we write $K = G\dot{\cup}H$ where $|G| = |J|, |H| = |K|$. Write $\{v'_k\} = \{t_g\} \cup \{t_h\}$ and rewrite β as follow:

$$\beta = \begin{pmatrix} z_t & w'_i & w'_l & u'_m & v'_n & t_g & t_h \\ 0 & w'_i\beta & w'_l\beta & u'_m\beta & v'_n\beta & t_g\beta & t_h\beta \end{pmatrix}.$$

As before, we can define $\lambda, \mu \in S(V,W)$ such that $\alpha = \lambda\beta\mu$. $\square$

The following example shows that in general $\mathcal{D} \neq \mathcal{J}$ on $S(V,W)$.

Example 1. Let $V = \langle w, w_k, u_1, u_2\rangle$ and $W = \langle w, w_k\rangle$, where K is infinite. Choose $k_0 \in K$ and let $K' = K\backslash\{k_0\}$. Then we define

$$\alpha = \begin{pmatrix} u_1 & u_2 & \{w, w_{k_0}\} & w_{k'} \\ 0 & w & w_{k_0} & w_{k'} \end{pmatrix} \text{ and } \beta = \begin{pmatrix} u_1 & \{u_2, w, w_{k_0}\} & w_{k'} \\ 0 & w_{k_0} & w_{k'} \end{pmatrix}.$$

Hence $\alpha, \beta \in S(V,W)$ and $V\alpha = \langle w, w_k\rangle$, $V\beta = \langle w_k\rangle$; $W\alpha = \langle w_k\rangle = W\beta$ and $V\alpha \cap W = \langle w, w_k\rangle$, $V\beta \cap W = \langle w_k\rangle$. So $\dim(V\alpha) = |K| = \dim(V\beta), \dim(W\alpha) = |K| = \dim(W\beta)$ and $\dim(V\alpha/(V\alpha \cap W)) = 0 = \dim(V\beta/(V\beta \cap W))$, thus $\alpha\mathcal{J}\beta$. Since $\dim((V\alpha \cap W)/W\alpha) = 1 \neq 0 = \dim((V\beta \cap W)/W\beta)$, we have α and β are not $\mathcal{D}$-related on $S(V,W)$.

Even $\dim(W)$ is finite, we still have $\mathcal{D} \neq \mathcal{J}$ on $S(V,W)$.

Example 2. Let $V = \langle w_1, w_2, w_3, u, u_k\rangle$ and $W = \langle w_1, w_2, w_3\rangle$, where K is infinite. Then we define

$$\alpha = \begin{pmatrix} \{u, w_1\} & w_2 & w_3 & u_k \\ 0 & w_1 & w_3 & u_k \end{pmatrix} \text{ and } \beta = \begin{pmatrix} w_1 & w_2 & w_3 & u & u_k \\ 0 & w_1 & w_3 & w_2 & u_k \end{pmatrix}.$$

Then $\alpha, \beta \in S(V,W)$ and $V\alpha = \langle w_1, w_3, u_k\rangle$, $V\beta = \langle w_1, w_2, w_3, u_k\rangle$; $W\alpha = \langle w_1, w_3\rangle = W\beta$ and $V\alpha \cap W = \langle w_1, w_3\rangle$, $V\beta \cap W = \langle w_1, w_2, w_3\rangle$. Thus $\dim(V\alpha) = |K| = \dim(V\beta), \dim(W\alpha) = 2 = \dim(W\beta)$ and $\dim(V\alpha/(V\alpha \cap W)) = |K| = \dim(V\beta/(V\beta \cap W))$, so $\alpha\mathcal{J}\beta$. But $\dim((V\alpha \cap W)/W\alpha) = 0 \neq 1 = \dim((V\beta \cap W)/W\beta)$, we have α and β are not $\mathcal{D}$-related on $S(V,W)$.

Theorem 3. *$\mathcal{D} = \mathcal{J}$ on $S(V,W)$ if and only if $W = V$ or $W = \{0\}$ or $\dim V$ is finite.*

Proof. If $W = V$ or $W = \{0\}$, then $S(V,W) = T(V)$ and thus $\mathcal{D} = \mathcal{J}$ by [1, Volume 1, exercise 2.2.6]. Now, assume that $\dim V$ is finite. Let $\alpha, \beta \in S(V,W)$ be such that $\alpha\mathcal{J}\beta$. So by Theorem 2, $\dim(V\alpha) = \dim(V\beta), \dim(W\alpha) = \dim(W\beta)$ and $\dim(V\alpha/(V\alpha \cap W)) = \dim(V\beta/(V\beta \cap W))$. Then we can write $W\alpha = \langle w_i\alpha\rangle$, $W\beta = \langle w'_i\beta\rangle$ and $V\alpha/(V\alpha \cap W) = \langle u_k\alpha + V\alpha \cap W\rangle$, $V\beta/(V\beta \cap W) = \langle u'_k\beta + V\beta \cap W\rangle$. Let $(V\alpha \cap W)/W\alpha = \langle v_j\alpha + W\alpha\rangle$ and $(V\beta \cap W)/W\beta = \langle v'_l\beta + W\beta\rangle$. Thus $V\alpha = \langle w_i\alpha\rangle \oplus \langle v_j\alpha\rangle \oplus \langle u_k\alpha\rangle$ and

$V\beta = \langle w_i'\beta\rangle \oplus \langle v_l'\beta\rangle \oplus \langle u_k'\beta\rangle$. So $|I| + |J| + |K| = \dim(V\alpha) = \dim(V\beta) = |I| + |L| + |K|$ is finite (since $\dim V$ is finite) and thus $|J| = |L|$, that is $\dim(V\alpha\cap W)/W\alpha) = \dim(V\beta\cap W)/W\beta)$. Therefore, $\alpha\mathcal{D}\beta$ and hence $\mathcal{D} = \mathcal{J}$.

Conversely, assume that $\mathcal{D} = \mathcal{J}$ on $S(V,W)$, and suppose on contrary that $\dim V$ is infinite and $\{0\} \neq W \subsetneq V$. We consider two cases:

Case 1 : $\dim W$ is finite. Let $V = \langle w_1, \ldots, w_n, u, u_k\rangle$ and $W = \langle w_1, \ldots, w_n\rangle$ $(n \geq 1)$, where K is infinite. Define $\alpha, \beta \in S(V,W)$ as follows:

$$\alpha = \begin{pmatrix} \{u, w_1\} & w_2 & \ldots & w_n & u_k \\ 0 & w_2 & \ldots & w_n & u_k \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} w_1 & w_2 & \ldots & w_n & u & u_k \\ 0 & w_2 & \ldots & w_n & w_1 & u_k \end{pmatrix}.$$

Thus $V\alpha = \langle w_2, \ldots, w_n, u_k\rangle$, $V\beta = \langle w_1, \ldots, w_n, u_k\rangle$; $W\alpha = \langle w_2, \ldots, w_n\rangle = W\beta$ and $V\alpha\cap W = \langle w_2, \ldots, w_n\rangle$, $V\beta\cap W = \langle w_1, \ldots, w_n\rangle$. Hence $\dim(V\alpha) = |K| = \dim(V\beta), \dim(W\alpha) = n - 1 = \dim(W\beta)$ and $\dim(V\alpha/(V\alpha\cap W)) = |K| = \dim(V\beta/(V\beta\cap W))$, so $\alpha\mathcal{J}\beta$. Since $\dim((V\alpha\cap W)/W\alpha) = 0 \neq 1 = \dim((V\beta\cap W)/W\beta)$, we have α and β are not $\mathcal{D}$-related on $S(V,W)$ by Theorem 1 and this contradicts the assumption.

Case 2 : $\dim W$ is infinite. Let $V = \langle w, w_k, u, u_i\rangle$ and $W = \langle w, w_k\rangle$, where K is infinite. Choose $k_0 \in K$ and let $K' = K\backslash\{k_0\}$. Define $\alpha, \beta \in S(V,W)$ as follows:

$$\alpha = \begin{pmatrix} u_i & \{w, w_{k_0}\} & w_{k'} & u \\ 0 & w_{k_0} & w_{k'} & w \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} u_i & \{u, w, w_{k_0}\} & w_{k'} \\ 0 & w_{k_0} & w_{k'} \end{pmatrix}.$$

Hence $V\alpha = \langle w, w_k\rangle$, $V\beta = \langle w_k\rangle$; $W\alpha = \langle w_k\rangle = W\beta$ and $V\alpha\cap W = \langle w, w_k\rangle$, $V\beta\cap W = \langle w_k\rangle$. Thus $\dim(V\alpha) = |K| = \dim(V\beta), \dim(W\alpha) = |K| = \dim(W\beta)$ and $\dim(V\alpha/(V\alpha\cap W)) = 0 = \dim(V\beta/(V\beta\cap W))$, so $\alpha\mathcal{J}\beta$. Since $\dim((V\alpha\cap W)/W\alpha) = 1 \neq 0 = \dim((V\beta\cap W)/W\beta)$, we have α and β are not $\mathcal{D}$-related on $S(V,W)$ by Theorem 1 and this leads to a contradiction. □

4 Ideals of $S(V,W)$

Let p be any cardinal number and let

$$p' = min\{q : q > p\}.$$

Note that p' always exists since the cardinals are well-ordered and when p is finite we have $p' = p + 1 =$ the successor of p.

To describe ideals of $S(V,W)$ for any vector space V and any subspace W of V, we let

$$\dim V = a, \;\; \dim W = b \;\; \text{and} \;\; \dim V/W = c.$$

In addition, for each cardinals r, s, t such that $1 \leq r \leq a', 1 \leq s \leq b'$ and $1 \leq t \leq c'$, define

$$S(r,s,t) = \{\alpha \in S(V,W) : \dim(V\alpha) < r, \dim(W\alpha) < s \text{ and } \dim(V\alpha/(V\alpha \cap W)) < t\}.$$

Then if $r = a', s = b'$ and $t = c'$, we have

$$S(r,s,t) = S(a',b',c') = S(V,W).$$

And if $r = b', s = b'$ and $t = 1$, then $S(r,s,t) = S(b',b',1) = \{\alpha \in S(V,W) : \dim(V\alpha) < b', \dim(W\alpha) < b'$ and $\dim(V\alpha/(V\alpha \cap W)) < 1\} = L(V,W) = \{\alpha \in T(V) : V\alpha \subseteq W\}$ which is an ideal of $S(V,W)$ [5, page 9].

We observe that: if $W = V$, then $\dim V = a = \dim W$ and $\dim V/W = 0$. Thus, $S(r,r,1) = \{\alpha \in S(V,W) : \dim(V\alpha) < r\} = \{\alpha \in T(V) : \dim(V\alpha) < r\}$ which is an ideal of $T(V)$.

Theorem 4. *The set $S(r,s,t)$ is an ideal of $S(V,W)$.*

Proof. Let $\alpha \in S(r,s,t)$ and $\lambda, \mu \in S(V,W)$. Then $\dim(V\alpha) < r$, $\dim(W\alpha) < s$ and $\dim(V\alpha/(V\alpha \cap W)) < t$. By using the same proof as given in Theorem 2, we have $\dim(V\lambda\alpha\mu) \leq \dim(V\alpha) < r$, $\dim(W\lambda\alpha\mu) \leq \dim(W\alpha) < s$ and $\dim(V\lambda\alpha\mu/(V\lambda\alpha\mu \cap W)) \leq \dim(V\alpha/(V\alpha \cap W)) < t$. Hence $\lambda\alpha\mu \in S(r,s,t)$. Therefore, $S(r,s,t)$ is an ideal of $S(V,W)$. □

We note that: if $r \leq u, s \leq v$ and $t \leq w$, then $S(r,s,t) \subseteq S(u,v,w)$. The following example shows that there is an ideal in $S(V,W)$ which is not of the form $S(r,s,t)$ and the set of ideals of $S(V,W)$ does not form a chain under the set inclusion.

Example 3. Let $V = \langle w_1, w_2, u_1, u_2 \rangle$ and $W = \langle w_1, w_2 \rangle$. Then $\dim V = 4, \dim W = 2$ and $\dim V/W = 2$. Since $S(3,3,1)$ and $S(4,2,2)$ are ideals of $S(V,W)$, we have $S(3,3,1) \cup S(4,2,2)$ is also an ideal of $S(V,W)$. Suppose that $S(3,3,1) \cup S(4,2,2) = S(\ell,m,n)$ for some $1 \leq \ell \leq 5, 1 \leq m \leq 3$ and $1 \leq n \leq 3$. Let

$$\alpha = \begin{pmatrix} w_1 & w_2 & u_1 & u_2 \\ 0 & w_1 & w_2 & u_2 \end{pmatrix}.$$

Then $\dim(V\alpha) = 3$, $\dim(W\alpha) = 1$ and $\dim(V\alpha/(V\alpha \cap W)) = 1$ and hence $\alpha \in S(4,2,2)$. If $\ell < 4$ or $n < 2$, then we have $\alpha \in S(4,2,2) \backslash S(\ell,m,n)$. If $m < 3$, there is

$$\beta = \begin{pmatrix} w_1 & \{w_2, u_1\} & u_2 \\ w_1 & w_2 & 0 \end{pmatrix} \in S(3,3,1) \backslash S(\ell,m,n)$$

since $\dim(V\alpha) = 2$, $\dim(W\alpha) = 2$ and $\dim(V\alpha/(V\alpha \cap W)) = 0$. Both cases contradict our supposition. So $\ell \geq 4, m \geq 3$ and $n \geq 2$. Consider

$$\delta = \begin{pmatrix} w_1 & w_2 & u_1 & u_2 \\ w_1 & w_2 & u_1 & 0 \end{pmatrix},$$

we have $\dim(V\alpha) = 3$, $\dim(W\alpha) = 2$ and $\dim(V\alpha/(V\alpha \cap W)) = 1$ and thus $\delta \in S(4,3,2)$ but $\delta \notin S(3,3,1) \cup S(4,2,2)$, so $S(3,3,1) \cup S(4,2,2) \neq S(r,s,t)$ for all $r \geq 4, s \geq 3$ and $t \geq 2$. Since $\alpha \in S(4,2,2) \backslash S(3,3,1)$ and $\beta \in S(3,3,1) \backslash S(4,2,2)$, we conclude that the set of ideals of $S(V,W)$ does not form a chain.

Lemma 5. *The set of ideals of $S(V,W)$ forms a chain under the set inclusion if and only if $V = W$.*

Proof. If $V = W$, then $S(V,W) = S(V,V) = T(V)$ and the ideals of $T(V)$ are precisely the set $\{\alpha \in T(V) : \dim(V\alpha) < r\}$ where $1 \leq r \leq (\dim V)'$. So, we see that the set of ideals forms a chain under the set inclusion. Conversely, assume that $W \subsetneq V$. Let $V = \langle w, w_j, u, u_i \rangle$ and $W = \langle w, w_j \rangle$ (I or J can be empty). Then $\dim V = |I| + |J| + 2, \dim W = |J| + 1$ and $\dim V/W = |I| + 1$. Consider

$$\alpha = \begin{pmatrix} w & \{u, u_i, w_j\} \\ w & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} u & \{u_i, w, w_j\} \\ u & 0 \end{pmatrix},$$

we have $\alpha \in S(2,2,1) \backslash S(2,1,2)$ and $\beta \in S(2,1,2) \backslash S(2,2,1)$. Thus the set of ideals of $S(V,W)$ does not form a chain. □

To obtain ideals of $S(V,W)$ we need the following notation. Let U be a non-zero subspace of $S(V,W)$. Define

$$K(U) = \{\alpha \in S(V,W) : \dim(V\alpha) \leq \dim(V\beta), \dim(W\alpha) \leq \dim(W\beta) \text{ and}$$
$$\dim(V\alpha/(V\alpha \cap W)) \leq \dim(V\beta/(V\beta \cap W)) \text{ for some } \beta \in U\}.$$

Then we see that $U \subseteq K(U)$, and $U_1 \subseteq U_2$ implies $K(U_1) \subseteq K(U_2)$.

Theorem 5. *The ideals of $S(V,W)$ are precisely the set $K(U)$ for some non-zero subspace U of $S(V,W)$.*

Proof. Assume that I is an ideal of $S(V,W)$. Let $\alpha \in K(I)$. Then $\dim(V\alpha) \leq \dim(V\beta), \dim(W\alpha) \leq \dim(W\beta)$ and $\dim(V\alpha/(V\alpha \cap W)) \leq \dim(V\beta/(V\beta \cap W))$ for some $\beta \in I$ and thus by Theorem 2 we have $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in S(V,W)$. Since $\beta \in I$ is an ideal of $S(V,W)$, it follows that $\alpha = \lambda\beta\mu \in I$, and that $K(I) \subseteq I$. Usually, we have $I \subseteq K(I)$. Therefore, $I = K(I)$.

Conversely, we prove that $K(U)$ is an ideal of $S(V,W)$. Let $\alpha \in K(U)$ and $\lambda, \mu \in S(V,W)$. Then $\dim(V\alpha) \leq \dim(V\beta), \dim(W\alpha) \leq \dim(W\beta)$ and $\dim(V\alpha/(V\alpha \cap W)) \leq \dim(V\beta/(V\beta \cap W))$ for some $\beta \in U$. As before, we have $\dim(V\lambda\alpha\mu) \leq \dim(V\alpha)$, $\dim(W\lambda\alpha\mu) \leq \dim(W\alpha)$ and $\dim(V\lambda\alpha\mu/(V\lambda\alpha\mu \cap W)) \leq \dim(V\alpha/(V\alpha \cap W))$. So $\dim(V\lambda\alpha\mu) \leq \dim(V\beta)$, $\dim(W\lambda\alpha\mu) \leq \dim(W\beta)$ and $\dim(V\lambda\alpha\mu/(V\lambda\alpha\mu \cap W)) \leq \dim(V\beta/(V\beta \cap W))$. Hence $\lambda\alpha\mu \in K(U)$ and therefore $K(U)$ is an ideal of $S(V,W)$. □

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