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Diagrams in the Category of Fischer Spaces

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Abstract

In the category of symplectic partial linear spaces all graphs are diagrams. In this note we prove that in the category of Fischer spaces the only graphs that are diagrams are the Dynkin diagrams A_n, D_n, E_6, E_7, E_8 and their natural generalizations $A_\infty, A_\infty^\infty, D_\infty$.

1 Introduction

The Coxeter group of a single laced Dynkin diagram Γ is a 3-transposition group (G, D) such that $\Gamma \subset D$, $\langle\Gamma\rangle = G$ and has the following universal property: if (G', D') is a 3-transposition group with $\Gamma \subset D'$ and $\langle\Gamma\rangle = G'$ then, with some additional restrictions, G and G' are isomorphic as 3-transposition groups. B. Fischer asked whether the Dynkin diagrams were the only ones with this universal property. Here we give a positive answer to this question in the setting of Fischer spaces.

In [C4] we introduced the concept of diagram in categories of partial linear spaces of order two. More precisely, in such category $\mathbf{C}$, a graph is a diagram for a space P if it is an induced graph of the collinearity graph of P that generates P with a suitable universal property in $\mathbf{C}$ (see definition 1).

Two natural questions arise: Which graphs are diagrams in $\mathbf{C}$ and which spaces have diagrams.

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B. Fischer in the finite case and J.I. Hall ([H1], [H2]) without this condition proved that all graphs are diagrams in the category of symplectic spaces.

We considered the second question: In [C1], using the known classification (see for example J.I. Hall [H1]) of the finite reduced symplectic spaces we gave explicit diagrams for each type; in [C2] we discard the condition of being reduced and in [C3], without using this classification but with a simple geometric classification, gave diagrams for all finite symplectic spaces. Finally in [C4] we proved the existence without any restrictions.

As stated in the abstract here we prove that in the category of not necessarily finite Fischer spaces the only graphs that are diagrams for spaces are the single laced Dynkin diagrams A_n, D_n, E_6, E_7, and E_8 and their natural generalizations $A_\infty, A_\infty^\infty$, and D_∞.

2 Diagrams

A partial linear space of order two is a set of points and a set of lines, each line consisting of three points and such that two lines are disjoint or they meet exactly in one point. Two points are called collinear if they are different points of the same line. A morphism is a map that sends lines onto lines. A subspace Q of a space P is a subset of points of P such that if two points of Q are in a line then the third point is also in Q. A plane in P is a subspace generated by two intersecting lines.

For our purposes, a graph is a set of vertices and a set of edges, each edge consisting of two vertices. The two vertices of an edge are called adjacent. A graph morphism is a map between vertices that sends adjacent vertices into adjacent vertices and non adjacent into non adjacent vertices or to the same point.

If P is a partial linear space we shall denote also with P its collinearity graph, i.e., the vertices are the points and the edges are the pairs of collinear points. Any set Γ of points of P has the induced graph structure.

Let $\mathbf{C}$ be a category of partial linear spaces of order two.

Definition 1. A graph Γ is a diagram for a space P in $\mathbf{C}$ if Γ is a subgraph of P that generates P and such that for any graph morphism $\varphi\colon \Gamma \to Q$ with Q in $\mathbf{C}$ there exists a morphism of partial linear spaces $\Phi\colon P \to Q$ in $\mathbf{C}$ such that Φ restricted to Γ is φ.

This definition could be generalize to partial linear spaces on any order.

A **Fischer space** is a partial linear space (of order two) whose planes are either (see figure 1) affine planes over $\mathbb{F}_3$ or dual affine planes over $\mathbb{F}_2$ (Aschbacher [A2, p. 92]). A **symplectic space** is a Fischer space with only dual affine planes. Here we consider only **connected** spaces, that is, spaces with connected collinearity graphs. See [A2] seccion 18.

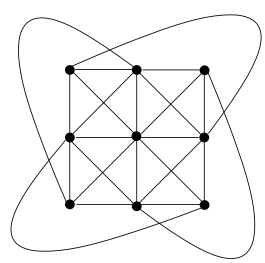
If we have a 3-transposition group (G, D) we associate a graph structure to D where the edges are the pairs (a, b) such that the order of ab is 3. ([A2] section 18).

Proposition 2. *Let Γ be a subgraph of a Fischer space P and $T(P)$ the 3-transposition group associated to P. (18.1 [A2]) The following are equivalent.*
a) Γ is a diagram for P.
b) for each 3-transposition group (G, D) with trivial center and for each morphism of graphs $f : \Gamma \to D$ there exists a unique map of groups $\tilde{f} : T(P) \to G$ such that $\tilde{f}_{|\Gamma} = f$.

Proof. This follows since there is a category equivalence between the category of Fischer spaces and the category of 3-transposition groups with trivial center given by sending P to $(T(P), L(T(P)))$. (see [A2] sec. 18.) □

Remark 3. *It is easy to see that a morphism of connected Fischer spaces sends non collinear points into non collinear points or to the same point.*

Remark 4. *From the previous remark we see that in Fischer spaces a morphism maps subspaces to subspaces. (see [A2] sec. 18).*



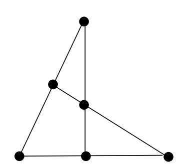


Figure 1: Affine plane over $\mathbb{F}_3$. Dual Affine plane over $\mathbb{F}_2$.

let **Fi** the category of Fischer spaces and **Sp** the subcategory of symplectic spaces.

Let Γ be a graph and $V = \mathbb{F}_2\Gamma$ be the vector space over the field $\mathbb{F}_2$ with basis Γ. Let $f \colon V \times V \to \mathbf{F_2}$ be the bilinear form given at the basis by $f(x, y) = 1$ if and only if x, y are adjacent. Let $q \colon V \to \mathbb{F}_2$ be the quadratic form given at the basis by $q(x) = 1$ and with associated form f. We call this form the quadratic form of the graph. Let $O(q)$ be the partial linear space with points x in V such that $q(x) = 1$ and lines the sets $\{x, y, z\}$ with $x + y + z = 0$.

Remark 5. *For any graph Γ there is a symplectic space P with Γ as an induced subgraph of the collinearity graph of P, for example $O(q)$ contains Γ.*

Lemma 6. *For a morphism $\Phi\colon P \to Q$ of Fischer spaces*

(i) *the image of an affine plane in P is an affine plane in Q.*

(ii) *the inverse image of an affine plane of Q (in the image of Φ) contains an affine plane in P.*

Proof. (i) Let Π be an affine plane in P. Now, $\Phi(\Pi)$ is a subspace with nine connected point therefore is an affine plane.

(ii) Let l'_1 and l'_2 be two intersecting lines of an affine plane Π' in $im(\Phi)$. Then we can easily find two different intersecting lines l_1 and l_2 in P whose images are l'_1 and l'_2, respectively. The plane generated in P by l_1 and l_2 can not be dual affine or the image of two non collinear points of P would be collinear in Q. □

Corollary 7. *If $\Phi\colon P \to Q$ is an epimorphism of Fischer spaces then P is symplectic if and only if Q is symplectic.*

Proposition 8. *In a full subcategory of* **Fi** *containing* **Sp** *the only spaces that may have diagrams are symplectic spaces.*

Proof. Let Γ be a diagram for a space P. From Remark 5 there exists a symplectic space Q and a morphism $\varphi\colon \Gamma \to Q$. Then, by Definition 1, there exists a morphism of Fischer spaces $\Phi\colon P \to Q$ such that $\Phi \,|\, \Gamma = \varphi$ and consequently, from Lemma 6, P can not have affine planes. □

Proposition 9. *Let Γ be a graph. If there exists a space P in a full subcategory* **C** *of* **Fi** *containing* **Sp** *such that $\Gamma \subset P$ and moreover the subspace generated by Γ in P has an affine plane, then Γ is not a diagram in* **C**.

Proof. If Γ is a diagram for a space Q in **C** then there exists a morphism from Q to the subspace generated by Γ in P. Then, from Lemma 6 (ii), Q would contain an affine plane in contradiction to Proposition 8. □

3 Spaces over $\mathbb{F}_3$

Let V be a vector space over the field $\mathbb{F}_3$ and let $f\colon V\times V\to\mathbb{F}_3$ be a bilinear symmetric form. Consider the projective space $\mathbf{P}(V)$ and the partial linear space $P=P(f)$ with the set of points

$$P=P(f)=\{[v]\in\mathbf{P}(V)\mid f(v,v)=1\}$$

and lines the sets $\ell=\{x,y,z\}$ of three points x,y,z in P such that

$$w=u+f(u,v)v \qquad (u\in x, v\in y, w\in z)\,.$$

Such spaces $P(f)$ will be called $\mathbb{F}_3$-spaces. Notice that if $x,y\in P$ then x and y are collinear if and only if $x\neq y$ and $f(u,v)\neq 0$ $(u\in x, v\in y)$.

For any $x\in P$ let γ_x be the permutation of the points of P given by

$$\gamma_x(y)=\begin{cases} y & \text{if } x \text{ and } y \text{ are not collinear}\\ z & \text{if } \{x,y,z\} \text{ is a line}\,. \end{cases} \tag{1}$$

From the definition of P we have that if $u\in x$, $v\in y$ then $\gamma_x(y)=[u+f(u,v)v]$.

Proposition 10. *The $\mathbb{F}_3$-spaces $P(f)$ are Fischer spaces.*

Proof. It is enough to check that for any $a\in P$, $\gamma_a\in\mathrm{Aut}(P)$ (see Aschbacher [A2, 18.1, p. 93]). Let $\{x,y,z\}$ be a line and $u\in x$, $v\in y$, $w\in z$, $r\in a$. Then we have that

$$\{\gamma_a(x),\gamma_a(y),\gamma_a(z)\}=\{[u+f(r,u)r],[v+f(r,v)r],[w+f(r,w)r]\}$$

and therefore $P(f)$ is a Fischer space (see Hall [H1, p. 101 (3.2)]). □

Let now Γ be a graph and let $V=\mathbb{F}_3\Gamma$ be the $\mathbb{F}_3$-vector space with basis the vertices of Γ. Define the symmetric bilinear form $f\colon V\times V\to\mathbb{F}_3$ given at the basis by

$$f(u,v)=\begin{cases} 1 & \text{if } u=v\\ 1 & \text{if } u \text{ and } v \text{ are adjacent}\\ 0 & \text{otherwise} \end{cases}$$

Let Q be the $\mathbb{F}_3$-space $P(f)$ given by this bilinear form. Then,

Remark 11. *Any graph Γ is an induced subgraph of an $\mathbb{F}_3$-space Q.*

4 Non-diagram graphs

In this section we give a set of obstructions for an $\mathbb{F}_3$-space to be symplectic and for a graph to be a diagram.

We denote with P_Γ the subspace of $Q = P(f)$ generated by Γ.

Proposition 12. *If Γ is one of six graphs in figure 2 then the space P_Γ contains an affine plane.*

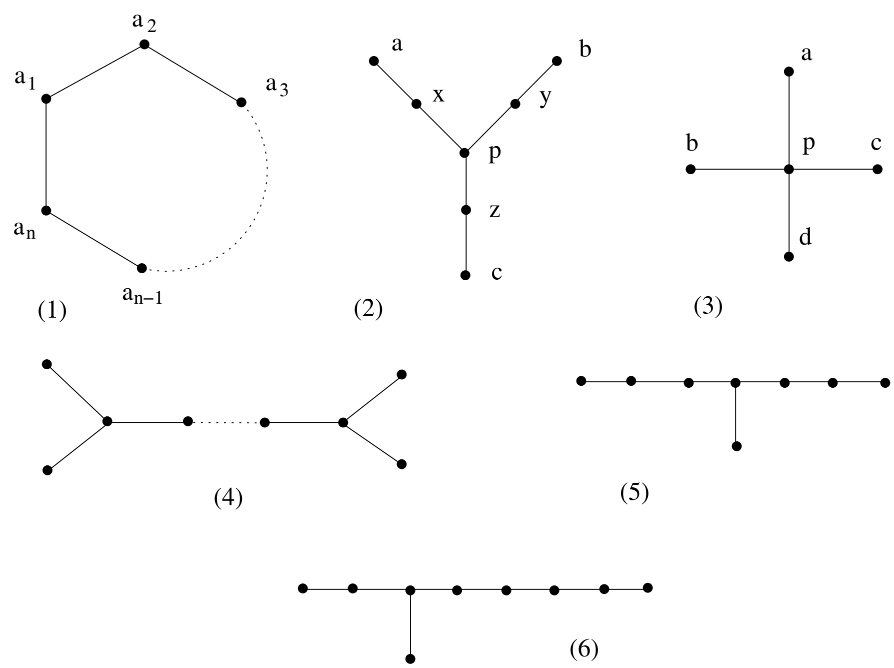


Figure 2:

Proof. In the first three cases we show in the space P_Γ a line ℓ and a point, not in the line ℓ, collinear with all three points of the line ℓ. Therefore P_Γ has an affine plane. In the first case we take the line $\{a-1, a_2+\cdots+a_n, a_1+a_2+\cdots+a_n\}$ then the point a_n is collinear with the three point of this line. In the second case the line $\{p, p+x+y+z, -p+x+y+z\}$ and the point $p+x+y+z+a+b+c$ solve the problem. In the third case $\{p, p+x+y+z, -p+x+y+z\}$ and the point w do the trick.

Thereafter we reduce the last three cases to the former ones, as in figure 3. $\square$

Let $\mathbf{C}$ be a full subcategory of $\mathbf{Fi}$ containing $\mathbf{Sp}$, all the $\mathbb{F}_3$-spaces and their respective subspaces.

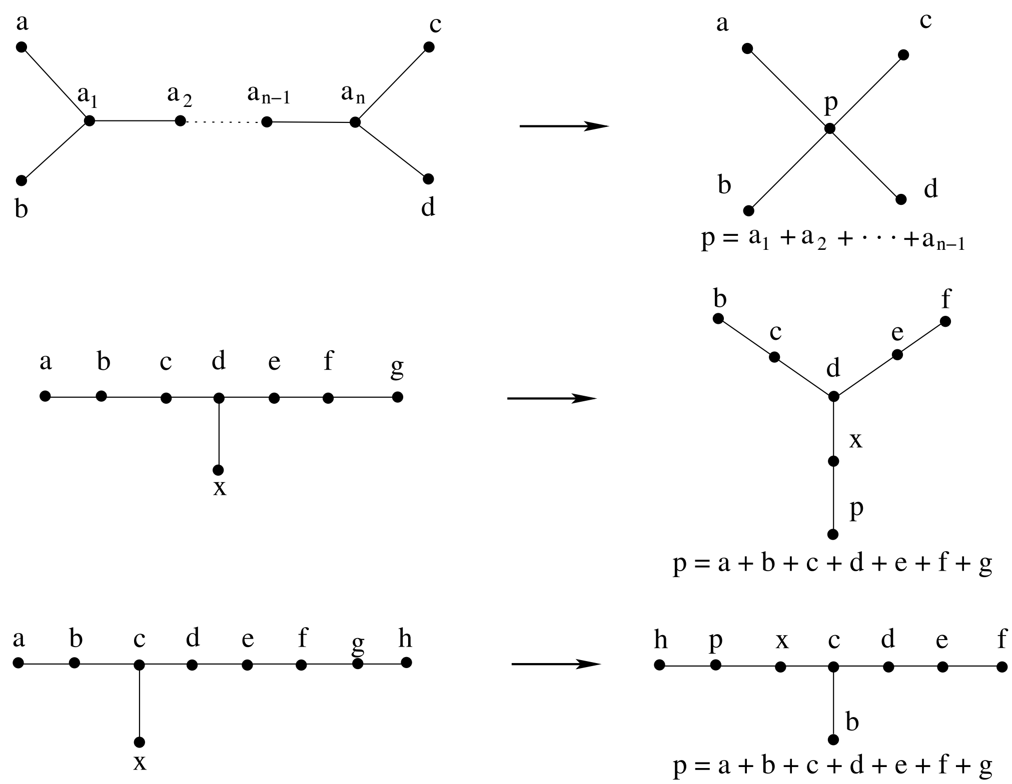


Figure 3:

Proposition 13. *If a graph* Γ *contains as an induced subgraph any of the six graphs in figure 2 then* Γ *is not a diagram in* $\mathbf{C}$.

Proof. Assume that Γ is a diagram in $\mathbf{C}$. Then there is a space P in $\mathbf{C}$ such that Γ is an induced subgraph of P generating P and with the corresponding universal property. Now, by remark 11, Γ is also an induced subgraph of an $\mathbb{F}_3$-space Q. Call $\varphi\colon \Gamma \to Q$ this immersion. Then there exists a morphism $\Phi\colon P \to Q$ in $\mathbf{C}$ such that $\Phi \,|\, \Gamma = \varphi$. Let Γ' be the induced subgraph of Γ in the figure 2. By Proposition 12 we have that the subspace $\langle \varphi(\Gamma') \rangle$ generated by Γ' in Q contains an affine plane Π. Therefore, using 4 in Lemma 6, we have

$$\Pi \subset \langle \varphi(\Gamma') \rangle \subset \langle \varphi(\Gamma) \rangle = \langle \Phi(\Gamma) \rangle \subset \Phi(\langle \Gamma \rangle) = \Phi(P)\,.$$

Then, again from Lemma 6, P contains an affine plane and so Γ cannot be a diagram in $\mathbf{C}$, (Proposition 8). $\square$

5 The diagrams in the category of Fischer spaces

Lemma 14. *If Γ is a connected graph that does not contains a graph of figure 2 (Proposition 12) then Γ is one of the following graphs:*

$$A_n,\ D_n,\ A_\infty,\ A_\infty^\infty,\ D_\infty,\ E_6,\ E_7,\ \text{and}\ E_8.$$

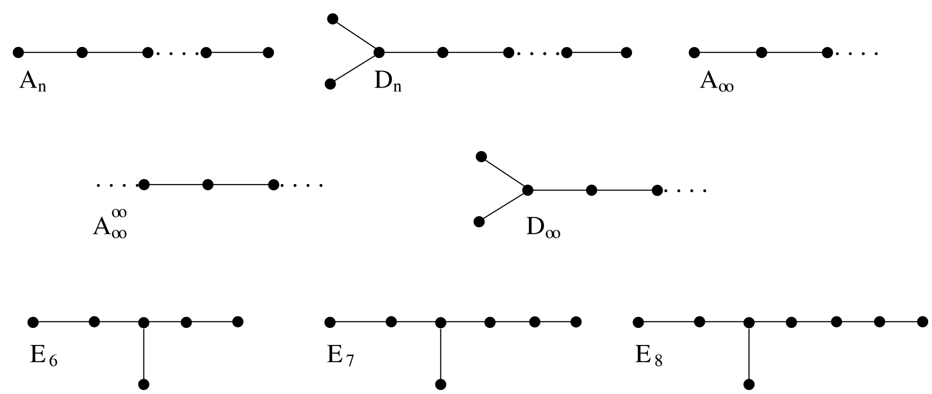


Figure 4:

Proof. Since Γ does not contain a graph of type (1), (3), or (4) then Γ is a tree with at most one vertex of valence three. If there is no such a vertex then Γ is A_n, A_∞ or A_∞^∞. If there is one valence three vertex then the obstructions (2), (5) and (6) forces Γ to be $D_n,\ D_\infty,\ E_6, E_7$ or E_8. □

As a consequence of Proposition 13 and Lemma 14 we have

Proposition 15. *In the category of Fischer spaces the only graphs that may be diagrams are*

$$A_n,\ D_n,\ A_\infty,\ A_\infty^\infty,\ D_\infty,\ E_6,\ E_7,\ \text{and}\ E_8\,.$$

Recall now that for a 3-transposition group (G, P), the conjugacy class P is a Fischer space (see Aschbacher [A2, 18.2 p. 94]).

Lemma 16. *Let Γ be a graph, G the Coxeter group of Γ and P the conjugacy class of Γ in G. If G is a 3-transposition group with 3-transposition class P then Γ is a diagram for P in* **Fi**.

Proof. Let Q be a Fischer space and let $\varphi\colon \Gamma \to Q$ be a graph morphism. Consider the group generated by the permutations (see equation 1 in section 3 and [A2] sec. 18).

$$T(Q) = \langle \gamma_a \mid a \in Q \rangle$$

and the injective map $\gamma\colon Q \to T(Q)$ given by $a \to \gamma_a$. Since G is a Coxeter group the map $\gamma\varphi$ gives a group morphism $\Psi\colon G \to T(Q)$ that restricted to Γ is $\gamma\varphi$.

$$\begin{tikzcd}
\Gamma \arrow[r] \arrow[d, "\varphi"'] & P \arrow[r] \arrow[dl, "\Phi"] & G \arrow[ddll, "\Psi"] \\
Q \arrow[d, "\gamma"'] & & \\
G_0(Q) & &
\end{tikzcd}$$

It is easy to see that Γ generates P as a Fischer space, therefore the restriction $\Phi = \Psi|P$ gives a morphism of Fischer spaces $\Phi\colon P \to Q$. Now since $\Phi|\Gamma = \varphi$ then $\Gamma \to P$ has the universal property of diagrams. Finally, since Γ generates P, Γ is a diagram for P. $\square$

Theorem 17. *In the category of Fischer spaces the only graphs that are diagrams are*

$$A_n,\ D_n,\ A_\infty,\ A_\infty^\infty,\ D_\infty,\ E_6,\ E_7,\ \textit{and}\ E_8\,.$$

Proof. By proposition 15, we just have to check that these graphs are diagrams in **Fi**. For that it is enough to see that the Coxeter groups of these graphs are indeed 3-transposition groups. For the graphs A_n and D_n. This is a well known property of the Weyl groups A, D and E. The infinite case follows easily. $\square$

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