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Groups with $\mathbb{Z}$-Torsion Homology

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Abstract

In this article we define the class of groups G with the property that there exists a natural number n_0 such that $H_n(G, M)$ is Z-torsion for every natural number $n \geq n_0$ and ZG-module M.

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1 Introduction

Let G be a group and M a left $\mathbb{Z}G$-module, then

$$H_n(G, M) = Tor_n^{\mathbb{Z}G}(\mathbb{Z}, M),$$

where $\mathbb{Z}$ is considered as a trivial $\mathbb{Z}G$-module. It is known (see [5, Th. 8.21]) that if R is a domain, then $Tor_n^R(A, B)$ is R-torsion for every R-modules A,B. It is natural then to consider questions concerning the property of being torsion for $H_n(G, M)$.

In the present article we study the following class of groups:

Definition 1.1. *Let $\mathcal{T}_H$ be the class of groups where $G \in \mathcal{T}_H$ if there exists a natural number n_0 such that $H_n(G, M)$ is $\mathbb{Z}$-torsionfor every $n \geq n_0$ and every $\mathbb{Z}G$-module M.*

It is obvious that $\mathcal{T}_H$ is not empty. For example finite groups [2, Prop. III.9.5, Cor. III.10.2] and groups with finite cohomological dimension belong to $\mathcal{T}_H$.

One of the main reasons for studying this class is the fact that it contains polycyclic by finite (Cor. 3.2) and locally finite groups (Th. 3.3). Thus a large number of groups belongs to $\mathcal{T}_H$. Furthermore, the notions of polycyclic and locally finite groups are central to group theory, hence every information concerning the homology of such groups is of interest.

It should be noted that interest for the class $\mathcal{T}_H$ may come from many other sources. For example Berrick and Kropholler in [1] study the reduced homology $\widetilde{H}(G) = \oplus_{n=1}^{+\infty} H_n(G, \mathbb{Z})$ of a group G with finite series whose factors are either infinite cyclic or locally finite. Their main result is that the reduced homology of such a group is infinite or zero. Furthermore it is proved that if $\widetilde{H}(G)$ is torsion then then G is locally finite and either:

1. for some prime p occuring as the order of an element of G, and for infinitely many n, $H_n(G, \mathbb{Z})$ contains elements of order p; or

2. G is acyclic.

In section 2 we study closure properties of the class $\mathcal{T}_H$. Examples of groups in the class are given in section 3.

2 Closure properties

In this section we present some of the closure properties of $\mathcal{T}_H$. More specifically $\mathcal{T}_H$ is closed under subgroups, amalgamated products, direct products and extensions.

Theorem 2.1. *The class $\mathcal{T}_H$ is closed under subgroups*

Proof. The proof is an immediate consequence of "Shapiro's lemma". In particular if H is a subgroup of a group G in $\mathcal{T}_H$, then there exists a natural number n_0 such that $H_n(G, M)$ is $\mathbb{Z}$-torsion for every $n \geq n_0$ and every $\mathbb{Z}G$-module M. From "Shapiro's lemma" for every natural number n and every $\mathbb{Z}H$-module M' we have the isomorphism [2, Prop. 6.2]:

$$H_n(H, M') \simeq H_n(G, Ind_H^G M').$$

Hence, $H_n(H, M')$ is $\mathbb{Z}$-torsion for every $n \geq n_0$ and every $\mathbb{Z}H$-module M'. □

Theorem 2.2. *Let $G = G_1 *_A G_2$ be the amalgamated product of G_1 and G_2 along A with G_1, G_2, A in $\mathcal{T}_H$. Then G belongs to $\mathcal{T}_H$.*

Proof. Since G_1, G_2, A belong to $\mathcal{T}_H$ there exists natural numbers n_0 such that $H_n(G_1, M)$, $H_n(G_2, M)$, $H_n(A, M)$ are $\mathbb{Z}$-torsion for every $n \geq n_0$ and every $\mathbb{Z}G$-module M. The result follows from the long exact sequence

$$\cdots \to H_k(A, M) \to H_k(G_1, M) \oplus H_k(G_2, M) \to H_k(G, M) \to H_{k-1}(A, M) \to \cdots$$

since for $k \geq n_0 + 1$ both $H_k(G_1, M) \oplus H_k(G_2, M)$ and $H_{k-1}(A, M)$ are $\mathbb{Z}$-torsion. □

Theorem 2.3. *Let $G = G_1 \times G_2$ be the direct product of G_1 and G_2 with G_1, G_2 in $\mathcal{T}_H$. Then G belongs to $\mathcal{T}_H$.*

Proof. Since G_1, G_2 belong to $\mathcal{T}_H$ there exist natural numbers n_1, n_2 such that $H_n(G_1, M)$ is $\mathbb{Z}$-torsion for $n \geq n_1$ and $H_n(G_2, M)$ is $\mathbb{Z}$-torsion for $n \geq n_2$ and every $\mathbb{Z}G$-module M. If $n \geq n_1 + n_2 - 1$, then $H_p(G_1, M) \otimes H_q(G_2, M)$ is $\mathbb{Z}$-torsion for every p, q with $p + q = n$. Hence from the Künneth formula

$$0 \to \bigoplus_{p+q=n} H_p(G_1, M) \otimes H_q(G_2, \mathbb{Z}) \to H_n(G_1 \times G_2, M) \to$$

$$\bigoplus_{p+q=n-1} Tor_1^{\mathbb{Z}}(H_p(G_1, \mathbb{Z}), H_q(G_2, M)) \to 0$$

and the fact that $Tor_1^{\mathbb{Z}}(H_p(G_1, M), H_q(G_2, M))$ is $\mathbb{Z}$-torsion, it follows that $H_n(G_1 \times G_2, M)$ is $\mathbb{Z}$-torsion. □

Theorem 2.4. *If there exists an extension*

$$0 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0$$

where H and G/H belong to $\mathcal{T}_H$, then G belongs to $\mathcal{T}_H$.

Proof. Since H, G/H belong to $\mathcal{T}_H$ there exist natural numbers n_1, n_2 such that $H_n(H, M)$ is $\mathbb{Z}$-torsion for every $n \geq n_2$, $H^n(G/H, M)$ is $\mathbb{Z}$-torsion for every $n \geq n_2$ and every $\mathbb{Z}G$-module M. The short exact sequence gives rise to a Lyndon - Hochschild -Serre spectral sequence [5, Th. 11.46]

$$E^2_{p,q} = H_p(G/H, H_q(H, M)) \Longrightarrow H_{p+q=n}(G, M).$$

It follows that we have a filtration

$$0 = F^0 H_n(G, M) \subseteq F^1 H_n(G, M) \subseteq \cdots \subseteq F^n H_n = H_n(G, M)$$

where $F^i H_n(G, M)/F^{i-1} H_n(G, M)$, $i = 0, \ldots, n$ is a subquotient of $E^2_{i,n-i} = H_i(G/H, H_{n-i}(H, M))$.

From the we hypothesis we have that $E^2_{p,q} = H_p(G/H, H_q(H, M))$ is $\mathbb{Z}$-torsion if $n = p + q \geq n_1 + n_2 - 1$. Hence $H_n(G, \mathbb{Z})$ is $\mathbb{Z}$-torsion for every $n \geq n_0 + 1$. □

3 Examples of groups in $\mathcal{T}_H$

3.1 Almost polycyclic groups

We remind that group G is is called almost polycyclic if there exist a subnormal series

$$1 = G_0 \trianglelefteq \cdots \trianglelefteq G_n = G$$

where each quotient G_i/G_{i-1} is finite or cyclic. If every quotient is cyclic the group is called polycyclic.

A group G is called noetherian if every subgroup is finitely generated.

Theorem 3.1. *The class $\mathcal{T}_H$ contains almost polycyclic groups.*

Proof. The proof is based on induction on the series length. If G has series length 0 then it is the trivial group. Suppose that every almost polycyclic group H having a series

$$1 = H_0 \trianglelefteq \cdots \trianglelefteq H_n = H$$

where each quotient H_i/H_{i-1} is finite or cyclic is contained in $\mathcal{T}_H$. If G is an almost polycyclic group with series

$$1 = G_0 \trianglelefteq \cdots \trianglelefteq G_n \trianglelefteq G_{n+1} = G$$

where each quotient G_i/G_{i-1} is finite or cyclic, then G_n is almost polycyclic and belongs to $\mathcal{T}_H$ by the induction argument. From the short exact sequence

$$1 \to G_n \to G_{n+1} \to G_{n+1}/G_n \to 1$$

and theorem 2.4 it follows that G belongs to $\mathcal{T}_H$. □

Corollary 3.2. *Polycyclic by finite groups belong to $\mathcal{T}_H$.*

A theorem of Mal'cev states that every soluble group of the general linear group $GL(n, \mathbb{Z})$ is polycyclic [6, Th. 4.4]. It follows that every soluble subgroup of the general linear group of finite rank is contained in $\mathcal{T}_H$. More generally, every noetherian group of matrices of finite rank over any field is polycyclic by finite [7], hence such a group is contained in $\mathcal{T}_H$.

3.2 Locally finite groups

A group is called locally finite if every finitely generated subgroup is finite. A countable locally finite group can be seen as the direct limit of finite groups. The fact that homology commutes with direct limits enables us to show that countable locally finite groups belong to $\mathcal{T}_H$.

Theorem 3.3. *Leg G be a countable locally finite group. Then, G is contained in $\mathcal{T}_H$.*

Proof. If $G = \{g_0 = 1, g_1, \ldots\}$ is a countable locally finite group, let $G = \lim_{\rightarrow} G_i$ where G_i is the finite group generated by the elements $g_1, \ldots, g_i$. For every i and every $\mathbb{Z}G$-module M $H_n(G_i, M)$ is $\mathbb{Z}$-torsion if n is greater or equal than 1. Since $H_n(G, M)$ is isomorphic to $\lim_{\rightarrow} H_n(G_i, M)$, it follows that $H_n(G, M)$ is $\mathbb{Z}$-torsion for every $\mathbb{Z}G$-module M if n is greater or equal than 1. □

The class of elementary amenable groups is the smallest class containing abelian and finite groups and closed under groups extensions and direct unions. It is known that, torsion groups in the class of elementary amenable groups are locally finite [3]. Hence torsion elementary amenable groups belong to $\mathcal{T}_H$.

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