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Global Dimension of the Category of Diagrams in an Additive Category

Ayman Machhidan

Abdelmalek Essaadi University
Faculty of sciences of Tetouan
Department of Mathematics and informatics
Ayman.Mach@gmail.com

Abstract

In this paper we will present a study of relative global dimension of some categories of functors. Given a finite poset $\mathcal{C}$, an additive category $\mathcal{A}$ (with some properties), and an abelian structure $\mathcal{P}$. There is a first quadrant spectral sequence defined by the reduced cohomologies of open intervals in $\mathcal{C}$ with coefficients in the relative extension groups in $\mathcal{A}$.

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1 Introduction

The origins of relative homological algebra can be found in different branches of algebra but mainly in the theory of abelian groups. Buchsbaum [2] have given axioms for a "proper class" of short exact sequences in any abelian category and Mac Lane has rewritten in his "Homology" [6] a part of homological algebra from the point of view of relative homological algebra. The first years after that have yielded many interesting Examples of proper classes which have been used for proving "relative" versions of "absolute" theorems. Heller [4] considered additive categories with a distinguished class of "proper" morphisms.
This article is devoted to extension groups in the category of functors from a small category to an additive category with an abelian structure in the sense of Heller [4]. It is proved that under additional assumptions there exists a spectral sequence which converges to the extension groups.
In section 2 we introduce the notion of abelian structure in an additive category with cancellation axiom. This generalization of the usual notion of a proper class of short exact sequences in abelian category. Section 3 contains definitions

and classical results in the cohomology of small category. Our main purpose is to introduce the reduced cohomology of open intervals (in a poset) with coefficients in an abelian group. In section 4 we give the definitions and useful property of the relative extension group in the category of functor.
Section 5 contains some applications of the results of the preceding sections to the comparison of the global dimension of the category of functor to additive category $\mathcal{A}$ (under additional assumptions) and the global dimension of the category $\mathcal{A}$.

2 Abelian structure in additive category

2.1 The cancellation axiom

Definition 2.1 *Let $\mathcal{A}$ be a pre-additive category, $B \in \mathcal{A}$ an object. A direct sum decomposition (A, A', i, p, i', p') of B consist of objects and morphisms $A \xrightarrow{i} B \xleftarrow{i'} A'$, $A \xleftarrow{p} B \xrightarrow{p'} A'$ satisfying to the conditions :*

$$p \circ i = 1_A,\ p' \circ i' = 1_{A'},\ p' \circ i = 0_{AA'},\ p \circ i' = 0_{A'A}\ \text{, and } i \circ p + i' \circ p' = 1_B$$

A morphism $\rho : B \to A$ is called a retraction if there exists a morphism $\nu : A \to B$ such that $\rho \circ \nu = 1_A$.

Lemma 2.2 *Let $\mathcal{A}$ be an additive category. Then the following conditions are equivalent*

1. *Every retraction has a kernel;*
2. *For every morphisms $p : B \to A$ and $i : A \to B$ in $\mathcal{A}$ satisfying $p \circ i = 1_A$ there are an object A' and morphisms $p' : B \to A'$, $i' : A' \to B$ such that (A, A', i, p, i', p') is a direct sum decomposition of B.*

2.2 Abelian structure

Definition 2.3 *Let $\mathcal{A}$ be an additive category with Cancellation Axiom. An Abelian structure on $\mathcal{A}$ is a subclass $\mathcal{P} \subseteq Mor(\mathcal{A})$ whose elements are called to be proper maps. The class $\mathcal{P}$ is to be subject to the following axioms :*
(P_0). $1_A \in \mathcal{P}$ for all $A \in Ob(\mathcal{A})$.
(P_1). If $(f : B \to C) \in \mathcal{P}$, $(g : A \to B) \in \mathcal{P}$ and g is an epimorphism then $f \circ g \in \mathcal{P}$; dually, if $f, g \in \mathcal{P}$ and f is a monomorphism then $f \circ g \in \mathcal{P}$.
(P_2). If $f \circ g \in \mathcal{P}$ is a monomorphism then g is proper; dually, if $f \circ g \in \mathcal{P}$ is an epimorphism then f is proper.

(P_3). *For every proper map $f : B \to D$ there are proper s.e.s $0 \to A \to B \to C \to 0$ and $0 \to C \to D \to E \to 0$ such that the following diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle 1_C} & & \\ 0 & \longleftarrow & E & \longleftarrow & D & \longleftarrow & C & \longleftarrow & 0 \end{array}$$

is commutative.
(P_4). *If in the commutative diagram*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

all columns and the second two rows are proper s.e.s, then the first row is a proper s.e.s.
A short exact sequence whose maps are proper is a proper s.e.s.

Example 2.4 *The class of all short exact sequences in an additive category $\mathcal{A}$ with cancellation form an Abelian structure.*
We define a proper class $\mathcal{D}$ of short exact sequences in an abelian category as one satisfying self dual axioms, where we let $\mathcal{M}(\mathcal{D})$ and $\mathcal{E}(\mathcal{D})$ denote the classes of monomorphisms and epimorphisms respectively associated with short exact sequences in $\mathcal{D}$.

- *All co-retracts are in $\mathcal{M}(\mathcal{D})$ [and consequently all retracts are in $\mathcal{E}(\mathcal{D})$].*
- *If $\alpha, \beta \in \mathcal{M}(\mathcal{D})$ and $\beta\alpha$ is defined, then $\beta\alpha \in \mathcal{M}(\mathcal{D})$. Dually if $\gamma, \delta \in \mathcal{E}(\mathcal{D})$ and $\delta\gamma$ is defined, then $\delta\gamma \in \mathcal{E}(\mathcal{D})$.*
- *If $\beta\alpha \in \mathcal{M}(\mathcal{D})$ and β is a monomorphism, then $\alpha \in \mathcal{M}(\mathcal{D})$. Dually if $\delta\gamma \in \mathcal{E}(\mathcal{D})$ and γ is an epimorphism, then $\delta \in \mathcal{E}(\mathcal{D})$.*

Then the class $\mathcal{D}$ is an Abelian structure.

Let $\mathcal{C}$ a small category and $\mathcal{A}$ an additive category. Denote by $\mathcal{A}^{\mathcal{C}}$ the category of functors $\mathcal{C} \longrightarrow \mathcal{A}$ and natural transformation, call it the category of diagrams of objects in $\mathcal{A}$. It is well knew that if $\mathcal{A}$ is additive then $\mathcal{A}^{\mathcal{C}}$ is also additive.

Remark 2.5 *Let $\mathcal{A}$ be an additive category with cancellation Axiom, $\mathcal{C}$ a small category. If $\rho : F \to G$ is a retraction in $\mathcal{A}^{\mathcal{C}}$ then $\rho_c : F(c) \to G(c)$ are retractions for all $c \in Ob(\mathcal{C})$ and have kernels by Lemma. Consequently, $\rho : F \to G$ has a kernel. Therefore $\mathcal{A}^{\mathcal{C}}$ is an additive category with Cancellation axiom.*

Lemma 2.6 *Let $\mathcal{A}$ be an additive category with an abelian structure $\mathcal{P}$ and $\mathcal{C}$ a small category. Denote by $\mathcal{CP}$ the class of all natural transformations $\eta : F \to G$ for which $\eta_c \in \mathcal{P}$ for all $c \in \mathcal{C}$. Then the class $\mathcal{CP}$ is an abelian structure in $\mathcal{A}^{\mathcal{C}}$.*

Proof 1 *We proof the axioms $P_0 - P_4$ for $\mathcal{CP}$:*
(P_0) Each identity $1_F : F \to F$ consists of $1_{F(c)}$, $c \in Ob(\mathcal{C})$. Therefore $1_F \in \mathcal{P}$.
(P_1) A morphism $\eta : F \to G$ in $\mathcal{A}^{\mathcal{C}}$ is a monomorphism if and only η_c are monomorphisms for all $c \in Ob(\mathcal{C})$; dually for epimorphisms.
(P_2) Analogously.
(P_3) Let $(Kerf, f^k)$ the kernel of a proper morphism $f \in \mathcal{P}$. By the universal property of kernels, for every commutative diagram :

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A' & \xrightarrow[f']{} & B' \end{array}$$

with $f, f' \in \mathcal{P}$ and $\alpha, \beta \in Mor\mathcal{A}$ there exists unique morphism $kerf \to kerf'$ for which the following diagram is commutative

$$\begin{array}{ccccc} kerf & \xrightarrow{f^k} & A & \xrightarrow{f} & B \\ \downarrow & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ kerf' & \xrightarrow{f'^k} & A' & \xrightarrow[f']{} & B' \end{array}$$

Dually we choose cokernels $(cokerf, f^c)$.

(P_4) Component wise.

3 Cohomology of Small Categories

Henceforth Ab is the category of abelian groups and homomorphisms, $\mathbb{Z}$ is the abelian group of integers, and $lim_{\mathcal{C}}$: $Ab^{\mathcal{C}} \longrightarrow Ab$ is the limit functor. Let $\mathcal{C}$ be

a small category, $F : \mathcal{C} \to Ab$ a functor.
Consider the sequence of groups

$$C^n(\mathcal{C}, F) = \prod_{c_0 \to \cdots \to c_n} F(c_n), \quad n \geq 0$$

Let $N_n\mathcal{C}$ be the set of all sequences of morphisms $c_0 \to \cdots \to c_n \in \mathcal{C}$. Regarding each $\varphi \in \prod_{c_0 \to \cdots \to c_n} F(c_n)$ as a function $\varphi : N_n\mathcal{C} \to \bigcup_{c \in Ob(\mathcal{C})} F(c)$ with $\varphi(c_0 \to \cdots \to c_n) \in F(c_n)$, we define the homomorphisms $d^n : C^n(\mathcal{C}, F) \to C^{n+1}(\mathcal{C}, F)$ by the formulas

$$(d^n\varphi)(\, c_0 \xrightarrow{\alpha_1} c_1 \longrightarrow \cdots \xrightarrow{\alpha_{n+1}} c_{n+1}\,) = \sum_{i=1}^{n} (-1)^i \varphi(\, c_0 \xrightarrow{\alpha_1} \cdots \longrightarrow \hat{c}_i \xrightarrow{\alpha_{i+1}} \cdots \xrightarrow{\alpha_{n+1}} c_{n+1}\,)$$

$$+ (-1)^{n+1} F(\alpha_{n+1})(\varphi(\, c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} c_n\,))$$

Here $(\, c_0 \xrightarrow{\alpha_1} \cdots \longrightarrow \hat{c}_i \xrightarrow{\alpha_{i+1}} \cdots \xrightarrow{\alpha_{n+1}} c_{n+1}\,)$ is equal to
$(\, c_0 \xrightarrow{\alpha_1} \cdots \longrightarrow c_{i-1} \xrightarrow{\alpha_{i+1} \circ \alpha_i} c_{i+1} \longrightarrow \cdots \xrightarrow{\alpha_{n+1}} c_{n+1}\,)$ if $0 < i < n+1$, and to $(\, c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n+1}} c_{n+1}\,)$ if $i = 0$. it is well know that $d^{n+1} \circ d^n = 0$ for all $n \geq 0$. We have obtained the complex of abelian groups and homomorphism

$$0 \longrightarrow C^0(\mathcal{C}, F) \xrightarrow{d^0} \cdots \longrightarrow C^n(\mathcal{C}, F) \xrightarrow{d^n} C^{n+1}(\mathcal{C}, F) \longrightarrow \cdots$$

which is denoted by $C^*(\mathcal{C}, F)$. The cohomologies $H^n(C^*(\mathcal{C}, F)) = ker\, d^n / Im\, d^{n-1}$ are isomorphic to abelian groups $lim^n_{\mathcal{C}} F$ (see [8]) where $lim^n_{\mathcal{C}} : Ab^{\mathcal{C}} \to Ab$ are the n-th right derived functors of the limit functor $lim_{\mathcal{C}} : Ab^{\mathcal{C}} \to Ab$.
For each $c \in Ob(\mathcal{C})$ we denote by $c/\mathcal{C}$ the *comma-category* in the sense of [6], its objects are pairs (a, α) of $a \in Ob(\mathcal{C})$ and $\alpha \in \mathcal{C}(c, a)$, and for each pair of objects $(a, \alpha \in \mathcal{C}(c, a))$ and $(b, \beta \in \mathcal{C}(c, b))$ the set of morphisms from (a, α) to (b, β) consists of $\gamma \in \mathcal{C}(a, b)$ for which the following diagrams

$$\begin{array}{ccc} c & \xrightarrow{\alpha} & a \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \gamma} \\ c & \xrightarrow[\beta]{} & b \end{array} \qquad (1)$$

Let $Q_c : c/\mathcal{C} \to \mathcal{C}$ be the functor which carries the above diagram (1) to $\gamma : a \to b$. The category contains the initial object $(c, 1_c)$. Therefore, the functor $lim_{c/\mathcal{C}}$ is exact and $H^n(C^*(\mathcal{C}, GQ_c)) = 0$ for every functor $F : \mathcal{C} \to Ab$ and $n > 0$.
Let $\mathcal{C}$ be a small category. For each $a \in Ob(\mathcal{C})$ we identify the morphism 1_a with the object a, so $Ob(\mathcal{C}) \subseteq Mor(\mathcal{C})$. Objects of the *factorization category* [1] $\mathcal{C}'$

are all morphisms of $\mathcal{C}$, and the set of morphisms $\mathcal{C}'(f,g)$ for any $f,g \in Mor(\mathcal{C})$ consists of pairs (α,β) of morphisms in $\mathcal{C}$ for which the diagram

$$\begin{array}{ccc} b & \xrightarrow{\beta} & d \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ a & \xleftarrow[\alpha]{} & c \end{array}$$

is commutative. We denote morphisms by $(\alpha,\beta): f \to g$. The composition and identity are defined component-wise. Functors $F:\mathcal{C}' \to Ab$ are called *natural systems.*
Let $\mathcal{C}$ be a small category, $F:\mathcal{C}' \to Ab$ a natural system. For every $n \geq 0$ we consider an abelian group

$$K^n(\mathcal{C},F) = \prod_{c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} c_n} F(\alpha_n \circ \alpha_{n-1} \circ \cdots \circ \alpha_1)$$

We regard elements of $K^n(\mathcal{C},F)$ as maps $\varphi: N_n\mathcal{C} \to \bigcup_{g\in Mor(\mathcal{C})} F(g)$ with $\varphi(\alpha_1,\cdots,\alpha_n) \in F(\alpha_n \circ \alpha_{n-1} \circ \cdots \circ \alpha_1)$ and define homomorphisms $d^n: K^n(\mathcal{C},F) \to K^{n+1}(\mathcal{C},F)$ by the formula

$$(d^n\varphi)(\alpha_1,\cdots,\alpha_{n+1}) = F(\alpha_1,1)\varphi(\alpha_2,\cdots,\alpha_{n+1})+$$
$$\sum_{i=1}^{n}(-1)^i\varphi(\alpha_1,\cdots,\alpha_{i-1},\alpha_{i+1}\circ\alpha_i,\alpha_{i+2},\cdots,\alpha_{n+1})+$$
$$(-1)^{n+1}F(1,\alpha_{n+1})\varphi(\alpha_1,\cdots,\alpha_{n+1}).$$

Cohomology groups $H^n(K^*(\mathcal{C},F)) = kerd^n/Im(d^{n-1})$ are called n-th cohomology groups $H^n(\mathcal{C},F)$ of $\mathcal{C}$ with coefficients in the natural system F.
For any $\alpha \in Mor(\mathcal{C})$ we denote by $dom\,\alpha$ and $cod\,\alpha$ the domain and codomain.

We consider each partially ordered set (poset) $\mathcal{C}$ as a small category whose objects are elements in $\mathcal{C}$ and morphisms are pairs of elements $a \leq b$ in $\mathcal{C}$.

Proposition 3.1 *Consider $pt = [0]$ be a category consisting of one object 0 and one morphism 1_0. Then :*

$$H_n(pt) = \begin{cases} 0 & if\, n > 0 \\ \mathbb{Z} & if\, n = 0 \end{cases}$$

Proof 2 *For every $n \geq 0$ the abelian group $C_n(pt)$ is generated by the element X_n which is equal to the sequence of n identity morphisms* $0 \xrightarrow{1_0} 0 \xrightarrow{1_0} \cdots \xrightarrow{1_0} 0$ *. Since $d_i^n(X_n) = X_{n-1}$, thus $C_*(pt)$ is isomorphic to the complex*

$$0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{1} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{1} \cdots$$

which n-th homology groups are equal to 0 if $n > 0$, and $H_0(C_*(pt)) \cong \mathbb{Z}$. Then we obtain the desired result.

Definition 3.2 *Let pt be the poset that consist of a sole point, and A an abelian group. There is a unique mapping $f : \mathcal{C} \to pt$. It determines the homomorphisms $H^n(pt, A) \to H^n(\mathcal{C}, A)$. The reduced cohomology $\tilde{H}^n(\mathcal{C}, A)$ are the cokernels of these homomorphisms for $n \geq 0$.*
We put by Definition $\tilde{H}^{-1}(\varnothing, A) = A$ and $\tilde{H}^n(\varnothing, A) = 0$ if $n \neq -1$. For $n \leq -1$ we put $\tilde{H}^n(\mathcal{C}, A) = 0$.

Remark 3.3 *It is easy to see that $\tilde{H}^n(\mathcal{C}, A) \cong H^n(\mathcal{C}, A)$ for $n > 0$ and $\tilde{H}^0(\mathcal{C}, A)$ is the cokernel of the homomorphism $A \to lim_{\mathcal{C}} \Delta_{\mathcal{C}} A$ which relates to each $a \in A$ the thread of elements each of which equals a.*

Let $A\,[a \leq b] : \mathcal{C}' \longrightarrow Ab$ the functor which take the values :

$$A\,[a \leq b]\,(x \leq y) = \begin{cases} A\, for\; a = x, b = y \\ 0\; otherwise. \end{cases}$$

Lemma 3.4 *[5, Lemma 2.3]*
Let $\mathcal{C}$ a poset and A an abelian group. Then the following isomorphisms hold for all elements $a < b$ in $\mathcal{C}$:

$$lim^n_{\mathcal{C}'} A[a < b] \cong \tilde{H}^{n-2}(]a, b[\,, A),\; n \geq 0.$$

If $a = b$ then the groups $lim^n_{\mathcal{C}'} A[a \leq b]$ are zero for $n > 0$ and for $n = 0$ we have the equality $lim_{\mathcal{C}'} A[a \leq b] = A$.

4 Relative Extension Groups in the category of functor

4.1 Extension Groups

Definition 4.1
- *An object P is called $\mathcal{P}$-projective if for every proper epimorphism $\varepsilon : A \to B$ and for a morphism $\alpha : P \to B$ there is $\beta : P \to A$ such that $\varepsilon \circ \beta = \alpha$.*

- *Let $\mathcal{A}$ be an additive category with Cancellation Axiom, $\mathcal{P} \subseteq Mor(\mathcal{A})$ an abelian structure. We say that $\mathcal{A}$ has enough $\mathcal{P}$-projectives if for each $A \in Ob(\mathcal{A})$ there is a $\mathcal{P}$-projective object $P(A) \in \mathcal{A}$ and a proper epimorphism $P(A) \xrightarrow{\pi A} A$.*

- Let $\mathcal{A}$ be an additive category with an abelian structure $\mathcal{P}$, and with enough $\mathcal{P}$-projectives. Then for every $A \in Ob(\mathcal{A})$ we have some object $P(A)$ and some proper epimorphism $\pi A : P(A) \to A$.

- We denote by $\omega A : \Omega(A) \to P(A)$ a kernel of πA.
  Let $P_*(A)$ be the exact sequence of $\mathcal{P}$-projective objects and the proper morphisms which is obtained by sticking of sequences :

$$0 \longrightarrow \Omega(A) \xrightarrow{\omega A} P(A) \xrightarrow{\pi A} A \longrightarrow 0$$

$$0 \longrightarrow \Omega^2(A) \xrightarrow{\omega\Omega(A)} P(\Omega(A)) \xrightarrow{\pi\Omega(A)} \Omega(A) \longrightarrow 0$$

$$\cdots$$

$$0 \longrightarrow \Omega^{k+1}(A) \xrightarrow{\omega\Omega^k(A)} P(\Omega^k(A)) \xrightarrow{\pi\Omega^k(A)} \Omega^k(A) \longrightarrow 0$$

$$\cdots$$

That is $P_*(A)$ consists of morphisms and objects :

$$0 \longleftarrow P(A) \xleftarrow{\omega A \circ \pi\Omega(A)} P(\Omega(A)) \xleftarrow{\omega\Omega(A) \circ \pi\Omega^2(A)} P(\Omega^2(A)) \longleftarrow \cdots$$

$$\cdots \longleftarrow P(\Omega^k(A)) \xleftarrow{\omega\Omega^k(A) \circ \pi\Omega^{k+1}(A)} P(\Omega^{k+1}(A)) \longleftarrow \cdots$$

- Given arbitrary objects A and B of the category $\mathcal{A}$, let $Ext^n_{\mathcal{P}}(A, B)$ denote the relative extension groups. Heller proved that $Ext^n_{\mathcal{P}}(A, B) \cong H^n(\mathcal{A}(P_*(A), B)), \forall n \geq 0$.
  These groups define the bifunctors

$$Ext^n_{\mathcal{P}}(-, =) : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow Ab.$$

Lemma 4.2 *[5, Lemma 3.1]*
Suppose that $\mathcal{A}$ is an arbitrary pre-additive category. Then we have a natural isomorphism of abelian group (for $F, G \in \mathcal{A}^{\mathcal{C}}$) :

$$lim_{\mathcal{C}'}\{\mathcal{A}(F(dom\alpha), G(cod\alpha))\} \cong \mathcal{A}^{\mathcal{C}}(F, G)$$

Lemma 4.3 *Let $\mathcal{C}$ be a small category, $\mathcal{A}$ an additive category with coproducts, $F : \mathcal{C} \to \mathcal{A}$ a functor, $D = \{D_c\}_{c \in Ob(\mathcal{C})}$ a family of objects $D_c \in Ob(\mathcal{A})$. Then*

$$lim^n_{\mathcal{C}'}\{\mathcal{A}(\Lambda D(dom\alpha), F(cod\alpha))\} = 0, \quad \forall n > 0$$

Proof 3 *For every $A \in Ob(\mathcal{A})$ and a set E we denote by $\sum_E A$ the coproduct of the family $\{A_e\}_{e\in E}$ of objects $A_e = A$. Let $\mu_e : A \to \sum_{e\in E} A$ be the canonical injections. Consider the isomorphism $\omega : \mathcal{A}(\sum_E A, B) \to Ab(LE, \mathcal{A}(A,B))$ which assigns to every morphism $\varphi : \sum_E A \to B$ the homomorphism $\omega(E,A,B) : LE \to \mathcal{A}(A,B)$ acting on $e \in E$ as $e \to \varphi \circ \mu_e \in \mathcal{A}(A,B)$.*
Then we fix an arbitrary $c \in Ob(\mathcal{C})$. Consider the sets $E = h^c(A) = \mathcal{C}(c,a)$ with $a \in Ob(\mathcal{C})$. Let Lh^c be the composition of L and h^c. Then there is an isomorphism $\mathcal{A}(\sum_{\mathcal{C}(c,a)} A, B) \to Ab(Lh^c(a), h^A(B))$, which is natural in $a \in \mathcal{C}$ and $A, B \in \mathcal{A}$. Let $\Lambda^c : \mathcal{A} \to \mathcal{A}^{\mathcal{C}}$ be the left adjoint to the functor $\mathcal{A}^{\mathcal{C}} \to \mathcal{A}$ acting as $F \to F(c)$ for $F \in Ob(\mathcal{A}^{\mathcal{C}})$ and $\eta \to \eta_c$ for $\eta \in Mor(\mathcal{A}^{\mathcal{C}})$. It is know that $(\Lambda^c A)(a) = \sum_{\mathcal{C}(c,a)} A$ for all $a \in Ob(\mathcal{C})$.
For each functor $F : \mathcal{C} \to \mathcal{A}$ we have an isomorphism of bi-functors

$$\mathcal{A}(\Lambda^c A(-), F(=)) \cong Ab(Lh^c(-), (h^A \circ F)(=)).$$

It leads to an isomorphism

$$\mathcal{A}(\Lambda^c A(dom\alpha), F(cod\alpha)) \cong Ab(Lh^c(dom\alpha), (h^A \circ F)(cod\alpha)) \tag{2}$$

of natural system on $\mathcal{C}$ with fixed c. For every family $D = \{D_c\}_{c\in Ob(\mathcal{C})}$ of objects in $\mathcal{A}$ there exists an isomorphism $\Lambda D \cong \sum_{c\in\mathcal{C}} \Lambda^c D_c$. Hence

$$lim^n_{\mathcal{C}'}\{\mathcal{A}(\Lambda D(dom\alpha), F(cod\alpha))\} \cong \prod_{c\in\mathcal{C}} lim^n_{\mathcal{C}'}\{\mathcal{A}(\Lambda^c D_c(dom\alpha), F(cod\alpha))\}$$

Thus, by the isomorphism 2 it suffices to prove that

$$lim^n_{\mathcal{C}'}\{Ab(Lh^c(dom\alpha), (h^A \circ F)(cod\alpha))\} = 0$$

for every $A \in Ob(\mathcal{A})$ and $n > 0$.
We will prove that $lim^n_{\mathcal{C}'}\{Ab(Lh^c(dom\alpha), G(cod\alpha))\} = 0$, for every $c \in Ob(\mathcal{C})$ and $G \in Ab^{\mathcal{C}}$.
For that purpose we consider the natural system $M = \{Ab(Lh^c(dom\alpha), G(cod\alpha))\}$ and the complex $K^*(\mathcal{C}, M)$. Then

$$K^n(\mathcal{C}, M) = \prod_{c_0\to\cdots\to c_n} Ab(Lh^c(c_0), G(c_n)) \cong \prod_{c_0\to\cdots\to c_n} \prod_{c\to c_0} G(c_n)$$

We recall that $C^n(c/\mathcal{C}, GQ_c)$ consists of functions

$$g : N_n(c/\mathcal{C}) \longrightarrow \bigcup_{\alpha\in Ob(c/\mathcal{C})} GQ_c(\alpha)$$

with $g(c_0 \to \cdots \to c_n) \in GQ_c(c \to c_0)$, *the homomorphisms* $d^n : C^n(c/\mathcal{C}, GQ_c) \to C^{n+1}(c/\mathcal{C}, GQ_c)$ *act as*

$$(d^n g)(c \to c_0 \to \cdots \to c_{n+1}) = \sum_{i=0}^{n} (-1)^i g(c \to c_0 \to \cdots \to \hat{c}_i \to \cdots \to c_{n+1})$$

$$+ (-1)^{n+1} G(c_n \to c_{n+1}) g(c \to c_0 \to \cdots \to c_n).$$

We will prove that the complex $K^*(\mathcal{C}, M)$ *for* $M = \{Ab(Lh^c(dom\alpha), G(cod\alpha))\}$ *is isomorphic to* $C^n(c/\mathcal{C}, GQ_c)$. *It is clear that* $K^n(\mathcal{C}, M) \cong C^n(c/\mathcal{C}, GQ_c)$. *The homomorphism* $d^n : K^n(\mathcal{C}, M) \to K^{n+1}(\mathcal{C}, M)$ *has the following action on functions* f *for which* $f(c_0 \to \cdots \to c_n)(c \to c_0) \in G(c_n)$ *:*

$$d^n f(c_0 \to c_1 \to \cdots \to c_{n+1})(c \to c_0) = f(c_1 \to \cdots \to c_{n+1}) \circ Lh^c(c_0 \to c_1)(c \to c_0)$$

$$+ \sum_{i=1}^{n} (-1)^i f(c_0 \to \cdots \to \hat{c}_i \to \cdots \to c_{n+1})(c \to c_0)$$

$$+ (-1)^{n+1} F(c_n \to c_{n+1}) f(c_0 \to \cdots \to c_n)(c \to c_0)$$

We $\tilde{f}(c \to c_0 \to \cdots \to c_n) = f(c_0 \to \cdots \to c_n)(c \to c_0)$. *We check that the correspondence* $f \to \tilde{f}$ *is a morphism of complexes :*

$$d^n \tilde{f}(c_0 \to c_1 \to \cdots \to c_{n+1})(c \to c_0) = \tilde{f}(c \to \hat{c}_0 \to c_1 \to \cdots \to c_{n+1})$$

$$+ \sum_{i=1}^{n} (-1)^i \tilde{f}(c \to c_0 \to \cdots \to \hat{c}_i \to \cdots \to c_{n+1})$$

$$+ (-1)^{n+1} F(c_n \to c_{n+1}) \tilde{f}(c \to c_0 \to \cdots \to c_n)$$

Thus, the map $f \to \tilde{f}$ *is an isomorphism of complexes* $K^*(\mathcal{C}, M)$ *and* $C^*(c/\mathcal{C}, GQ_c)$. *Consequently, the* n*-th cohomology groups of* $K^*(\mathcal{C}, M)$ *are zeros for* $n > 0$. *Hence* $lim^n_{\mathcal{C}'}\{Ab(Lh^c(dom\,\alpha), G(cod\,\alpha))\} = 0$ *for all* $n > 0$. *The isomorphism (2) and preservation of products by* $lim^n_{\mathcal{C}'}$ *finish the proof.*

4.2 Spectral Sequences of Ext

- Suppose that $\mathcal{C}$ is a poset, $\mathcal{A}$ is an additive category, and $A \in \mathcal{A}$ is an object of $\mathcal{A}$. Given an element $c \in \mathcal{C}$, denote by $A\,[c] : \mathcal{C} \longrightarrow \mathcal{A}$ the functor that takes the values :

$$A\,[c]\,(x) = \begin{cases} A & for\ x = c; \\ 0 & otherwise. \end{cases}$$

Theorem 4.4 *(The Main Theorem).*
Suppose that $\mathcal{C}$ is a finite poset , $\mathcal{A}$ is an additive category with co-products , and $\mathcal{P}$ is a non empty abelian structure in $\mathcal{A}$, and $\mathcal{A}$ has enough $\mathcal{P}$-projectives. Then for arbitrary $A, B \in Ob(\mathcal{A})$ and elements $a < b$ in $\mathcal{C}$ there is a first-quadrant spectral sequence of the type :

$$E_2^{p,q} = \tilde{H}^{p-2}(]a,b[, Ext^q_{\mathcal{P}}(A,B)) \tag{3}$$

which converges to the graded abelian group $\{Ext^n_{\mathcal{CP}}(A\,[a]\,, B\,[b])\}_{n\geq 0}$

- In the present work we apply the spectral sequence (1) to studying the (relative) global dimension of the category of functors defined on a finite poset with values in an additive category.

Theorem 4.5 *Let $\mathcal{A}$ be an additive category with co-products, $\mathcal{P}$ a non-empty abelian structure. If the co-product of every family of proper epimorphisms is proper, and $\mathcal{A}$ has enough $\mathcal{P}$-projective, then for each small category $\mathcal{C}$ and functors $F, G : \mathcal{C} \to \mathcal{A}$ there exists a first quadrant spectral sequence of the type :*

$$E_2^{p,q} = lim^p_{\mathcal{C}'}\{Ext^q_{\mathcal{P}}(F(dom\alpha), G(cod\alpha))\} \qquad (2)$$

which converges to the graded abelian group $\{Ext^n_{\mathcal{CP}}(F,G)\}_{n\geq 0}$.

Proof 4 *We will build an exact sequence $0 \leftarrow F \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$ of proper morphisms with $\mathcal{CP}$-projective F_n for all $n \geq 0$. Recall that $O : \mathcal{A}^{\mathcal{C}} \to \mathcal{A}^{Ob(\mathcal{C})}$ is the restriction functor, and Λ the left adjoint to O. The co-unit of adjunction $\varepsilon_F : \Lambda OF \to F$ is a retraction on each $c \in Ob(\mathcal{C})$. We choose a family $P = \{P(c)\}_{c\in Ob(\mathcal{C})}$ of $\mathcal{P}$-projectives and a family of proper epimorphisms $\psi = \{\psi_c : P(c) \to F(c)\}_{c\in Ob(\mathcal{C})}$ and apply the functor Λ. Then $\Lambda(\psi) : \Lambda(P) \to \Lambda OF$ is a proper epimorphism as a co-product of proper epimorphisms.*
We let $F_0 = \Lambda(P)$. Let K be a kernel of the morphism $\varepsilon \circ \Lambda(\psi) : F_0 \to F$. We apply above building to K instead of F and obtain some functor K_0. We let $F_1 = K_0$. Then the inclusion $K \to F_0$ is a proper monomorphism in $\mathcal{A}^{\mathcal{C}}$. Consequently, the composition $F_1 \to K \to F_0$ is a proper morphism. We denote it by d_0. By induction we build the members $F_2, F_3, \cdots$ and morphisms $d_n : F_{n+1} \to F_n$.
Now we consider the complex : $\{K^n(\alpha)\} = \{\mathcal{A}(F_n(dom\alpha), G(cod\alpha))\}$ in the category $Ab^{\mathcal{C}'}$. By Grothendieck theorem there are two spectral sequences concerned with hyper co-homologies of the functor $lim_{\mathcal{C}'}$ with respect to the complex K^* convergent to the same limit.

$$H^p(lim^q_{\mathcal{C}'}\{K^*(\alpha)\}) \quad ; \qquad lim^p_{\mathcal{C}'}\{H^q(K^*(\alpha))\}$$

For each $n \geq 0$ there exists a family $A = \{A_c\}_{c \in Ob(\mathcal{C})}$ of objects of A such that $F_n = \Lambda A$. Applying the Lemma 4.3 we obtain $lim^q_{\mathcal{C}'}\{K^n(\alpha)\} = 0$ for all $q > 0$, and $n \geq 0$. Thus, the first spectral sequence degenerates and second gives the looking spectral sequence.

Definition 4.6 *The Baues-Wirsching dimension of a small category $\mathcal{C}$ is*

$$dim(\mathcal{C}) = c.d(\mathcal{C}') = sup\{n \in \mathbb{N} : lim^n_{\mathcal{C}'} \neq 0\}.$$

Corollary 4.7 *Let $\mathcal{C}$ be a finite category of dimension $p = dim(\mathcal{C}) < \infty$, let $\mathcal{A}$ be an additive category with co-products, $\mathcal{P}$ a non-empty abelian structure such that the co-product of every family of proper epimorphisms is proper, and $\mathcal{A}$ has enough $\mathcal{P}$-projective.*
If $q = gl.dim_{\mathcal{P}}(\mathcal{A}) < \infty$, then for every functors $F, G : \mathcal{C} \to \mathcal{A}$ we have the following isomorphism

$$lim^p_{\mathcal{C}'}\{Ext^q_{\mathcal{P}}(F(dom\,\alpha), G(cod\,\alpha))\} \cong Ext^{p+q}_{\mathcal{C}\mathcal{P}}(F, G)$$

5 Relative global dimension of the category of diagrams

Definition 5.1 *The global dimension of a category $\mathcal{A}$ with respect to an abelian structure $\mathcal{P}$ is*

$$gl.dim_{\mathcal{P}}(\mathcal{A}) \;=\; Sup\{n \in \mathbb{N}|\; Ext^n_{\mathcal{P}}(-,=) \neq 0\}$$

Put $gl.dim_{\mathcal{P}}(\mathcal{A}) = -1$ if there are no such numbers $n \geq 0$. The absolute global dimension $gl.dim(\mathcal{A})$ of $\mathcal{A}$, is recovered when $\mathcal{P}$ consists of all short exact sequences in $\mathcal{A}$.

Lemma 5.2 *Suppose that $\mathcal{C}$ is a finite poset, $\mathcal{A}$ is an additive category, and $\mathcal{P}$ is an abelian structure in $\mathcal{A}$. Then $gl.dim_{\mathcal{C}\mathcal{P}}\,(\mathcal{A}^{\mathcal{C}})$ is the upper bound of the numbers n for which there exist $a, b \in \mathcal{C}$ and $A, B \in \mathcal{A}$ such that $Ext^n_{\mathcal{C}\mathcal{P}}(A\,[a]\,, B\,[b]) \neq 0$*

Proof 5 *Let $\mathcal{P}$ an abelian structure and $A \in \mathcal{A}$ an object, we define the relative projective dimension $pd_{\mathcal{P}}\,(A) \;=\; sup\,\{n \in \mathbb{N}\;/Ext^n_{\mathcal{P}}(A,-) \;\neq\; 0\}$*
If $gl.dim_{\mathcal{C}\mathcal{P}}(\mathcal{A}^{\mathcal{C}}) = n$ then there are functor $F, G \in \mathcal{A}^{\mathcal{C}}$ such that $pd_{\mathcal{C}\mathcal{P}}\,(F) = n$ and $Ext^n_{\mathcal{C}\mathcal{P}}(F, G) \neq 0$.
Suppose that $c \in \mathcal{C}$ is the minimal among such that $F(c) \neq 0$. We denote by F' the kernel of the epimorphism $F \to F(c)[c]$. We apply the functor $\mathcal{A}^{\mathcal{C}}(-, G)$

to $0 \longrightarrow F' \longrightarrow F \longrightarrow F(c)[c] \longrightarrow 0$*, we obtain a long exact sequence which contain the interval*

$$Ext^n_{\mathcal{CP}}(F(c)[c], G) \to Ext^n_{\mathcal{CP}}(F, G) \to Ext^n_{\mathcal{CP}}(F', G) \to 0$$

$a \in \mathcal{C}$ *such that :* $Ext^n_{\mathcal{CP}}(F(a)[a], G) \neq 0$*.*
We suppose now $c \in \mathcal{C}$ *is the maximal among such that* $G(c) \neq 0$*. We denote by* G'' *the cokernel of the natural transformation* $G(c)[c] \to G$*. Then* $0 \longrightarrow G(c)[c] \longrightarrow G \longrightarrow G'' \longrightarrow 0$ *is exact. Since* $Ext^n_{\mathcal{CP}}(F(a)[a], -)$ *is right exact, we obtain the exact :*

$$Ext^n_{\mathcal{CP}}(F(a)[a], G(c)[c]) \to Ext^n_{\mathcal{CP}}(F(a)[a], G) \to Ext^n_{\mathcal{CP}}(F(a)[a], G'') \to 0$$

We conclude $\exists b \in \mathcal{C}$ *such that :* $Ext^n_{\mathcal{CP}}(F(a)[a], G(b)[b]) \neq 0$*. Substitute* $F(a) = A$ *and* $G(b) = B$*.*

Corollary 5.3 *Suppose that* $\mathcal{C}$ *is a finite poset,* $\mathcal{A}$ *is an additive category, and* $\mathcal{P}$ *is a non empty abelian structure in* $\mathcal{A}$ *such that* $gl.dim_{\mathcal{P}}(\mathcal{A}) < \infty$*, and* $\mathcal{A}$ *has enough* $\mathcal{P}$*-projective.*
If the co-product of every family of proper epimorphisms is proper. Then : $gl.dim_{\mathcal{CP}}(\mathcal{A}^{\mathcal{C}}) = gl.dim_{\mathcal{P}}(\mathcal{A})$ *if and only if the poset* $\mathcal{C}$ *is discrete.*

Proof 6 *If* $\mathcal{C}$ *is not discrete then there exist elements* $a < b$ *in* $\mathcal{C}$ *such that* $]a, b[\, = \varnothing$*. In this case take objects such that* $A, B \in \mathcal{A}$ *:* $Ext^q_{\mathcal{P}}(A, B) \neq 0$ *then:*

$$Ext^{q+1}_{\mathcal{CP}}(A\,[a]\,, B\,[b]) \cong \tilde{H}^{-1}(]a, b[\,, Ext^q_{\mathcal{P}}(A, B)) = Ext^q_{\mathcal{P}}(A, B) \neq 0$$

Hence $gl.dim_{\mathcal{CP}}(\mathcal{A}^{\mathcal{C}}) \geq 1 + gl.dim_{\mathcal{P}}(\mathcal{A})$ *For the converse implication :*

$$Ext^q_{\mathcal{CP}}(F, G) \cong \prod_{c \in \mathcal{C}} Ext^q_{\mathcal{P}}(F(c), G(c))$$

Corollary 5.4 *Let* $\mathcal{C}$ *be a finite poset,* $\mathcal{A}$ *is an additive category with co-products ,* $\mathcal{P}$ *is a non empty abelian structure in* $\mathcal{A}$ *such that* $gl.dim_{\mathcal{P}}(\mathcal{A}) < \infty$ *and* $\mathcal{A}$ *has enough* $\mathcal{P}$*-projective. If the co-product of every family of proper epimorphisms is proper. Then*

$$gl.dim_{\mathcal{CP}}(\mathcal{A}^{\mathcal{C}}) = 1 + gl.dim_{\mathcal{P}}(\mathcal{A})$$

if and only if $\mathcal{C}$ *is not discrete and for every pair* $a < b$ *from* $\mathcal{C}$*, the subset* $]a, b[\subseteq \mathcal{C}$ *is totally ordered.*

Proof 7 *Suppose that the intervals* $[a, b]$ *are totally ordered. Then* $\tilde{H^n}(]a, b[, G) = 0; \forall n \geq 0$*. So we have* $E_2^{p,q} = 0; p \geq 2$*, and consequently* $Ext^n_{\mathcal{CP}}(A\,[a]\,, B\,[b]) = 0$*, for all* $n \geq 2 + gl.dim_{\mathcal{P}}\, \mathcal{A}$*, and* $A, B \in Ob\mathcal{A}$*. Hence* $gl.dim_{\mathcal{CP}}(\mathcal{A}^{\mathcal{C}}) \leq 1 + gl.dim_{\mathcal{P}}(\mathcal{A})$*.*
Conversely, suppose that $\exists a, b \in \mathcal{C}$ *for which* $]a, b[$ *is not totally ordered.* $H^i(]a, b[, G) = 0$ *for* $i > 0$*, and* $\tilde{H}^0(]a, b[, G) \neq 0$*. We obtain :*

$$Ext^{2+q}_{\mathcal{CP}}(A\,[a]\,, B\,[b]) \cong \tilde{H}^0(]a, b[, Ext^q_{\mathcal{P}}(A, B)) \neq 0$$

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