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Full Expletive Fuzzy Languages

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Abstract

A word $u \in A^+$ is an expletive for a fuzzy language λ if $\lambda(xuy) = \lambda(xy)$ for all $x, y \in A^+$. A full expletive fuzzy language is a fuzzy language in which the set of expletives is A^+. Here we prove that the set of full expletive fuzzy languages is a variety and the associated pseudovariety of semigroups is that of midunit semigroups.

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1 Introduction

In this section we recall the basic definitions, results and notations that will be used in the sequel. All undefined terms are as in [1, 3]. Let A be finite alphabet and let $A^*(A^+)$ be the free monoid(semigroup) over A. A language L is a subset of $A^*(A^+)$. L is recognizable if there exists a finite monoid(semigroup) M and a homomorphism $\varphi : A^* \to M$ ($\varphi : A^+ \to M$) such that $L = P\varphi^{-1}$ where $P \subseteq M$. Syntactic monoid(semigroup) is the smallest monoid(semigroup) recognizing L and if L is regular (accepted by a finite automaton), then the syntactic monoid (semigroup) is finite (cf. [3]).

A word $u \in A^+$ is called expletive for a language L if $xuy \in L \Leftrightarrow xy \in L$ for all $x, y \in A^+$. The set of all expletives for L is denoted by $Exp(L)$. A language L is called an expletive language if $L = Exp(L')$ for some language L'. L is called full expletive if $Exp(L) = A^+$. The syntactic semigroup of expletive language contains midunits and the set of full expletive languages in A^+ is a +-variety of languages and the associated pseudovariety of semigroups is that of midunit semiroups(cf.[4]).

A fuzzy language λ in A^* (A^+) is a fuzzy subset of A^* (A^+). Since the characteristic function χ_L of a language L is a fuzzy language, every language is a fuzzy language. To each fuzzy language λ we associate a congruence P_λ, called syntactic congruence, as follows.

For $u, v \in A^*(u, v \in A^+)$

$$uP_\lambda v \text{ if } \lambda(xuy) = \lambda(xvy) \text{ for all } x, y \in A^*(x, y \in A^+).$$

The quotient monoid(semigroup) $Syn(\lambda) = A^*/P_\lambda$ ($Syn(\lambda) = A^+/P_\lambda$) is called the syntactic monoid (semigroup) of λ and the homomorphism $\eta_\lambda : A^* \to Syn(\lambda)(\eta_\lambda : A^+ \to Syn(\lambda))$ is called the syntactic morphism of λ. Syntactic monoid(semigroup) is the smallest monoid(semigroup) recognizing λ(cf. [2]).

For fuzzy languages $\lambda, \lambda_1, \lambda_2$ over an alphabet A, complement, union and intersection are defined respectively by $\overline{\lambda}(u) = 1 - \lambda(u)$, $(\lambda_1 \vee \lambda_2)(u) = \lambda_1(u) \vee \lambda_2(u)$ and $(\lambda_1 \wedge \lambda_2)(u) = \lambda_1(u) \wedge \lambda_2(u)$ Further left and right quotients are defined respectively by

$$(\lambda_1^{-1}\lambda_2)(u) = \bigvee_{v \in A^*} (\lambda_2(vu) \wedge \lambda_1(v)) \text{ and } (\lambda_2\lambda_1^{-1})(u) = \bigvee_{v \in A^*} (\lambda_2(uv) \wedge \lambda_1(v))$$

Let $c \in [0, 1]$ be arbitrary. Then the fuzzy language $c\lambda$, called multiplication by a constant c, is defined by $(c\lambda)(u) = c \cdot \lambda(u)$. Let A, B be finite alphabets, $\phi : A^* \to B^*(\phi : A^+ \to B^+)$ be a homomorphism and ψ a fuzzy language in $B^*(B^+)$, then the inverse image of ψ (under ϕ) is a fuzzy language $\psi\phi^{-1}$ over A defined by $(\psi\phi^{-1})(u) = \psi(u\phi)$. For a fuzzy language λ by a c-cut, $c \in [0, 1]$, we mean the language λ_c defined by $\lambda_c = \{u \in A^* \ : \ \lambda(u) \geq c\}$.

A fuzzy language is regular if it is recognized by a fuzzy automaton. The following theorem gives a characterization for regular fuzzy languages.

Theorem 1.1 ([1]). *A fuzzy language λ is regular if and only if $Im(\lambda)$ is finite and language λ_c is regular for every $c \in [0, 1]$.*

Definition 1.2. *A family $\mathrm{F} = \mathrm{F}(A)$ of regular fuzzy languages is a variety of fuzzy languages in $A^*(A^+)$ if it is closed under unions, intersections, complements, multiplication by constants, quotients, inverse homomorphic images and cuts.*

For a variety of fuzzy languages F, let F^s be the family of finite monoids defined by $\mathrm{F}^s = \{\mathrm{Syn}(\lambda) \ : \ \lambda \in \mathrm{F}(A) \text{ for some } A\}$. For a variety of finite monoids S, let $\mathrm{S}^f = \mathrm{S}^f(A)$ be the family of fuzzy languages defined by $\mathrm{S}^f(A) = \{\lambda \text{ is a fuzzy language over } A \ : \ \mathrm{Syn}(\lambda) \in \mathrm{S}\}$.

Theorem 1.3 ([2], Theorem 7). *The mappings $\mathrm{F} \to \mathrm{F}^s$ and $\mathrm{S} \to \mathrm{S}^f$ are mutually inverse lattice isomorphisms between the lattices of all varieties of fuzzy languages and all varieties of finite monoids.*

2 Expletive fuzzy languages

Let λ be a fuzzy language in A^+. A word $u \in A^+$ is called an expletive for λ if $\lambda(xuy) = \lambda(xy)$ for all $x, y \in A^+$. The set of all expletives for λ is denoted by $E(\lambda)$. Thus

$$E(\lambda) = \{u \in A^+ \ : \ \lambda(xuy) = \lambda(xy) \text{ for all } x, y \in A^+\}.$$

Note that E defines a map from the set of all fuzzy languages in A^+ into the set of all languages in A^+.

Example 2.1. *Let $c \in [0,1]$ and let $\lambda^c : A^+ \to [0,1]$ be given by $\lambda(u) = c$ for all $u \in A^+$. Then $E(\lambda^c) = A^+$.*

Theorem 2.2. *Let λ be a fuzzy language in A^+. Then $E(\lambda)$ is a subsemigroup of A^+.*

Proof. Let $u, v \in E(\lambda)$. Then for all $x, y \in A^+$, we have $\lambda(x(uv)y) = \lambda(xy)$. So $uv \in E(\lambda)$ and hence $E(\lambda)$ is a subsemigroup of A^+. □

Theorem 2.3. *Let $L \subseteq A^+$. Then $E(\chi_L) = Exp(L)$*

Proof. Let $u \in E(\chi_L)$. Then $\chi_L(xuy) = \chi_L(xy)$ for all $x, y \in A^+$. So $xuy \in L$ if and only if $xy \in L$ for all $x, y \in A^+$. Thus $E(\chi_L) = Exp(L)$.

Theorem 2.4. *Let $u \in A^+$. Then $u \in E(\lambda)$ if and only if $[u]_\lambda$ is a midunit in $Syn(\lambda)$, where $[u]_\lambda$ denotes the P_λ class containing u.*

Proof. Let $u \in E(\lambda)$. Then $\lambda(xuy) = \lambda(xy)$ for all $x, y \in A^+$. So $\lambda(x_1(xuy)x_2) = \lambda(x_1(xy)x_2)$ for all $x_1, x_2 \in A^+$. Then $(xuy)P_\lambda(xy)$. That is $[x]_\lambda[u]_\lambda[y]_\lambda = [x]_\lambda[y]_\lambda$ for all $x, y \in A^+$. Thus $[u]_\lambda$ is a midunit in $Syn(\lambda)$.

Conversely, assume that $[u]_\lambda$ is a midunit in $Syn(\lambda)$. Then, by the similar arguements, $\lambda(xuy) = \lambda(xy)$ for all $x, y \in A^+$. Thus $u \in E(\lambda)$.

Definition 2.5. *A fuzzy language λ in A^+ is called an expletive fuzzy language if $E(\lambda) \neq \emptyset$.*

The fuzzy language given in Example 2.1 is an expletive fuzzy language. The following result is an immediate consequence of Theorem 2.4.

Corollary 2.6. *Let λ be an expletive fuzzy language in A^+. Then $Syn(\lambda)$ contains midunits.*

Theorem 2.7. *Let $u \in E(\lambda)$ and $uP_\lambda v$. Then $v \in E(\lambda)$.*

Proof. Since $\lambda(xuy) = \lambda(xy)$ and $\lambda(xuy) = \lambda(xvy)$ for all $x, y \in A^+$, we have $\lambda(xvy) = \lambda(xy)$ for all $x, y \in A^+$. Thus $v \in E(\lambda)$.

Corollary 2.8. *If λ is a regular fuzzy language, then $E(\lambda)$ is a recognizable language in A^+.*

Proof. By Theorem 2.7, $E(\lambda)\eta_\lambda\eta_\lambda^{-1} = E(\lambda)$. Also $Syn(\lambda)$ is finite by Theorem 1.1. Thus $E(\lambda)$ is recognizable.

3 Variety of Full Expletive Fuzzy Languages

Definition 3.1. *A regular fuzzy language is called full expletive if* $E(\lambda) = A^+$*.*

The set of all full expletive fuzzy languages over A is denoted by FEF (or $FEF(A)$). Since constant functions on A^+(Example 2.1) and χ_L(Theorem 2.3) for an expletive language are in FEF, it follows that $FEF \neq \emptyset$.

Lemma 3.2. *Let* $\lambda \in FEF$*. Then* $\overline{\lambda} \in FEF$*.*

Proof. Since $\overline{\lambda}(xuy) = \overline{\lambda}(xy)$ if and only if $\lambda(xuy) = \lambda(xy)$ for all $x, y, u \in A^+$, it follows that $E(\lambda) = E(\overline{\lambda})$. Since $E(\lambda) = A^+$, we have $\overline{\lambda} \in FEF$.

Lemma 3.3. *Let* $\lambda_1, \lambda_2 \in FEF$*. Then* $\lambda_1 \vee \lambda_2$ *and* $\lambda_1 \wedge \lambda_2$ *are in* FEF*.*

Proof. Let $u \in A^+$. Then $\lambda_1(xuy) = \lambda_1(xy)$ and $\lambda_2(xuy) = \lambda_2(xy)$ for all $x, y \in A^+$. Therefore

$$\begin{aligned}(\lambda_1 \vee \lambda_2)(xuy) &= \lambda_1(xuy) \vee \lambda_2(xuy) = \lambda_1(xy) \vee \lambda_2(xy)\\ &= (\lambda_1 \vee \lambda_2)(xy) \text{ for all } x, y \in A^+.\end{aligned}$$

Hence $(\lambda_1 \vee \lambda_2) \in FEL$. Since $\lambda_1 \wedge \lambda_2 = \overline{(\overline{\lambda_1} \vee \overline{\lambda_2})}$, we have $\lambda_1 \wedge \lambda_2 \in FEF$.

Lemma 3.4. *FEF is closed under multiplication by constants.*

Proof. Let $u \in A^+$ and $c \in [0, 1]$. Then for all fuzzy languages λ, we have

$$\begin{aligned}u \in E(\lambda) &\Leftrightarrow \lambda(xuy) = \lambda(xy) \text{ for all } x, y \in A^+\\ &\Leftrightarrow c\lambda(xuy) = c\lambda(xy) \text{ for all } x, y \in A^+\\ &\Leftrightarrow (c\lambda)(xuy) = (c\lambda)(xy) \text{ for all } x, y \in A^+ \Leftrightarrow u \in E(c\lambda)\end{aligned}$$

That is $E(\lambda) = E(c\lambda)$. Thus, if $\lambda \in FEF$, then $c\lambda \in FEF$.

Lemma 3.5. *FEF is closed under c-cuts.*

Proof. Let $\lambda \in FEF$ and $c \in [0, 1]$ and let $\lambda_c = \{u \in A^+ : \lambda(u) \geq c\} = L$. If $u \in A^+$, then for all $x, y \in A^+, xuy \in L \Leftrightarrow c \leq \lambda(xuy) = \lambda(xy) \Leftrightarrow xy \in L$. So $u \in \text{Exp}(L)$ and hence $\text{Exp}(L) = A^+$. By Theorem2.3, $E(\chi_L) = Exp(L) = A^+$. So FEF is closed under c-cuts.

Lemma 3.6. *FEF is closed under inverse homomorphic image.*

Proof. Let X be a finite alphabet, $\phi : X^+ \to A^+$ be a homomorphism and $\lambda \in FEF$. Then the inverse image of λ under ϕ $(\lambda\phi^{-1})$ is a fuzzy language over X. Let $u \in X^+$. Then for all $x, y \in X^+$, we have

$$(\lambda\phi^{-1})(xuy) = \lambda((xuy)\phi) = \lambda(x\phi u\phi y\phi) = \lambda(x\phi y\phi) = \lambda((xy)\phi) = \lambda\phi^{-1}(xy)$$

So $E(\lambda\phi^{-1}) = X^+$. Hence FEF is closed under inverse homomorphic images.

Lemma 3.7. *FEF is closed under quotients.*

Proof. Let $\lambda_1, \lambda_2 \in FEF$ and $u \in A^+$. Then for all $x, y \in A^+$ we have

$$\begin{aligned}(\lambda_1^{-1}\lambda_2)(xuy) &= \bigvee_{v \in A^+} \{\lambda_2(vxuy) \wedge \lambda_1(v)\} = \bigvee_{v \in A^+} \{\lambda_2((vx)uy) \wedge \lambda_1(v)\} \\ &= \bigvee_{v \in A^+} \{\lambda_2(vxy) \wedge \lambda_1(v)\} = (\lambda_1^{-1}\lambda_2)(xy)\end{aligned}$$

So $u \in E(\lambda_1^{-1}\lambda_2)$ implies that $E(\lambda_1^{-1}\lambda_2) = A^+$. Similarly $E(\lambda_2\lambda_1^{-1}) = A^+$. Thus $\lambda_1^{-1}\lambda_2 \in FEF$ and $\lambda_2\lambda_1^{-1} \in FEF$.

Theorem 3.8. *FEF is a variety of fuzzy languages.*

Proof. Follows from Lemmas 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7.

By Theorem 2.4, we have the following.

Theorem 3.9. *Let λ be a fuzzy language. Then $\lambda \in FEF$ if and only if Syn(λ) is a midunit semigroup.*

Theorem 3.10. *Let A fixed finite alphabet and let S be the finite variety of finite midunit semigroups. Then F_A^S is S and S^f is FEF.*

Proof. By Theorem 3.9 there exists a one-one correspondence between elements of F_A^S and that of S. Since FEF is a variety of fuzzy languages, the result follows by Theorem 1.3.

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