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Semi Hopfian and Semi Co-Hopfian Modules over Generalized Power Series Rings

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Abstract

Let $(S, \leq)$ be a strictly totally ordered monoid, R be a commutative ring and M be an R-module. We show the following results: (1) If $(S, \leq)$ satisfies the condition that $0 \leq s$ for all $s \in S$, then the module $[[M^{S,\leq}]]$ of generalized power series is a semi Hopfian $[[R^{S,\leq}]]$-module if and only if M is a semi Hopfian R-module; (2) If $(S, \leq)$ is artinian, then the generalized inverse polynomial module $[M^{S,\leq}]$ is a semi co-Hopfian $[[R^{S,\leq}]]$-module if and only if M is a semi co-Hopfian R-module.

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1 Introduction

Let R be a ring and $(S, \leq)$ be a strictly totally ordered monoid. Assume that $[[R^{S,\leq}]]$ is the ring of generalized power series with coefficients in R and exponents in S. The generalized inverse polynomial modules $[M^{S,\leq}]$ and the generalized power series modules $[[M^{S,\leq}]]$ are two important classes of modules over $[[R^{S,\leq}]]$, which have been studied by many authors, see for example [3, 4, 5, 6, 7, 8, 9, 16, 17]. The motivation for the paper comes from the following results. Recall that a module $_RM$ is called Hopfian (resp. co-Hopfian) if any surjective (resp. injective) endomorphism of $_RM$ is an isomorphism. Note that any noetherian module is Hopfian, and any artinian module is co-Hopfian. Let R be an associative ring not necessarily containing an identity element, it was shown in[16, Theorem 4.6] that if M is a noetherian left R-module possessing property (F), $(S, \leq)$ is narrow, S is cancellative, torsion-free and

$S = \langle s_1, \ldots, s_n \rangle + G(S)$ for some finite set $\{s_1, \ldots, s_n\}$ of elements in $S \setminus G(S)$, then $[[M^{S,\leq}]]$ is a noetherian left $[[R^{S,\leq}]]$-module, and that for a nonzero R-module M possessing property (F), if $[[M^{S,\leq}]]$ is noetherian, then so is M; if S is cancellative then there exist a finite number $s_1, \ldots, s_n$ of elements in $S \setminus G(S)$ satisfying $S = \langle s_1, \ldots, s_n \rangle + G(S)$; if $0 \leq s$ for all $s \in S$, then $(S, \leq)$ is narrow ([16, Proposition 2.1, Theorem 2.2]), where $G(S)$ denotes the largest subgroup of S and $\langle s_1, \ldots, s_n \rangle$ the submonoid of S generated by $s_1, \ldots, s_n$. It was shown in [3, Theorem 4] that if $(S, \leq)$ is a strictly totally ordered monoid which is finitely generated and satisfies the condition that $0 \leq s$ for any $s \in S$, then $[[M^{S,\leq}]]$ is a Hopfian left $[[R^{S,\leq}]]$-module if and only if M is a Hopfian left R-module. Let R be an associated ring not necessarily with identity, M a left R-module having the property (F), and $(S, \leq)$ a strictly totally ordered monoid which is also artinian, it was proven in [17, Theorem 4] that if S is a finitely generated monoid then $[M^{S,\leq}]$ is an artinian left $[[R^{S,\leq}]]$-module if and only if M is an artinian left R-module, and in [8, Theorem 2.2], it was shown that $[M^{S,\leq}]$ is a co-Hopfian left $[[R^{S,\leq}]]$-module if and only if M is a co-Hopfian left R-module. As a generalization of the notions Hopficity and co-Hopficity, semi Hopfian modules and semi co-Hopfian modules were introduced in [1] and a new characterization of artinian rings was obtained by using these concepts. Let R be a commutative ring and M an R-module, following [1], M is said to be *semi Hopfian* (resp. *semi co-Hopfian*) if for any $r \in R$, the endomorphism of M induced by multiplication by r is an isomorphism, provided it is surjective (resp. injective). Clearly, over a commutative ring R, any Hopfian (resp. co-Hopfian) R-module is semi-Hopfian (resp. semi co-Hopfian). In [2], it was shown that M is semi Hopfian R-module if and only if $M[X_1, \ldots, X_n]$ is semi Hopfian $R[X_1, \ldots, X_n]$-module if and only if $M[[X_1, \ldots, X_n]]$ is semi Hopfian $R[[X_1, \ldots, X_n]]$-module (Theorem 3.4), and that M is semi co-Hopfian R-module if and only if $M[X_1^{-1}, \ldots, X_n^{-1}]$ is semi co-Hopfian $R[X_1, \ldots, X_n]$-module if and only if $M[X_1^{-1}, \ldots, X_n^{-1}]$ is semi co-Hopfian $R[[X_1, \ldots, X_n]]$-module (Theorem 3.5), where $X_1, \ldots, X_n$ are n commutating indeterminates over R. Motivated by these facts, in this paper, the semi Hopficity of generalized power series modules and the semi co-Hopficity of generalized inverse polynomial modules will be investigated, respectively. To be precise, let $(S, \leq)$ be a strictly totally ordered monoid, R a commutative ring with identity and M an R-module, we will prove that if $(S, \leq)$ satisfies the condition that $0 \leq s$ for all $s \in S$, then the module $[[M^{S,\leq}]]$ of generalized power series is a semi Hopfian $[[R^{S,\leq}]]$-module if and only if M is a semi Hopfian R-module, and that if $(S, \leq)$ is artinian, then the generalized inverse polynomial module $[M^{S,\leq}]$ is a semi co-Hopfian $[[R^{S,\leq}]]$-module if and only if M is a semi co-Hopfian R-module.

2 Definitions and Notations

In this Section, we recall some definitions and notations, any concept and notation not defined here can be found in [3, 7, 14, 15, 16].

Let $(S, \leq)$ be an ordered set. Recall that $(S, \leq)$ is *artinian* if every strictly decreasing sequence of elements of S is finite, and that $(S, \leq)$ is *narrow* if every subset of pairwise order-incomparable elements of S is finite. Let S be a commutative monoid. Unless stated otherwise, the operation of S shall be denoted additively, and the neutral element by 0. The following definition is duo to P. Ribenboim [14, 15].

Definition 2.1 *Let $(S, \leq)$ be a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that, if $s, s', t \in S$ and $s < s'$, then $s+t < s'+t$), and R a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f : S \longrightarrow R$ such that $supp(f) = \{s \in S \mid f(s) \neq 0\}$ is artinian and narrow. For every $s \in S$ and $f, g \in [[R^{S,\leq}]]$, let $X_s(f, g) = \{(u, v) \in S \times S \mid u + v = s, f(u) \neq 0, g(v) \neq 0\}$. It follows from [14, 4.1] that $X_s(f, g)$ is finite. This fact allows to define the operation of convolution:*

$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)g(v).$$

With this operation, and pointwise addition, $[[R^{S,\leq}]]$ becomes a ring, which is called the ring of generalized power series with coefficients in R and exponents in S.

Examples and basic properties of rings of generalized power series are given in [14, 15].

Definition 2.2 *Let M be a left R-module and $(S, \leq)$ a strictly ordered monoid. We denote by $[[M^{S,\leq}]]$ the set of all maps $\phi : S \longrightarrow M$ such that $supp(\phi)$ is artinian and narrow, where $supp(\phi) = \{s \in S \mid \phi(s) \neq 0\}$. For each $s \in S$, $f \in [[R^{S,\leq}]]$ and $\phi \in [[M^{S,\leq}]]$, let $X_s(f, \phi) = \{(u, v) \in S \times S \mid u + v = s, f(u) \neq 0, \phi(v) \neq 0\}$. Then by [3, Lemma 1], $X_s(f, \phi)$ is finite. This allows to define the scalar multiplication as follows:*

$$(f\phi)(s) = \sum_{(u,v)\in X_s(f,\phi)} f(u)\phi(v).$$

With this operation and pointwise addition, $[[M^{S,\leq}]]$ becomes a left $[[R^{S,\leq}]]$-module, which is called the module of generalized power series with coefficients in M and exponents in S.

Similarly, if M is a right R-module, then $[[M^{S,\leq}]]$ is a right $[[R^{S,\leq}]]$-module. Examples and some results of modules of generalized power series are given in [3, 4, 6, 9, 16].

Definition 2.3 *Let M be a left R-module, we let $[M^{S,\leq}]$ be the set of all maps $\phi : S \longrightarrow M$ such that the set supp$(\phi) = \{s \in S | \phi(s) \neq 0\}$ is finite. Now $[M^{S,\leq}]$ can be turned into a left $[[R^{S,\leq}]]$-module under some additional conditions. The addition in $[M^{S,\leq}]$ is componentwise and the scalar multiplication is defined as follows:*

$$(f\phi)(s) = \sum_{t \in S} f(t)\phi(s+t), \quad \text{for every } s \in S,$$

where $f \in [[R^{S,\leq}]]$, and $\phi \in [M^{S,\leq}]$. Since the set supp(ϕ) is finite, this multiplication is well-defined. If $(S, \leq)$ is a strictly totally ordered monoid which is also artinian, then, from [7], $[M^{S,\leq}]$ becomes a left $[[R^{S,\leq}]]$-module, which we call the generalized inverse polynomial module.

Similarly, if M is a right R-module, then $[M^{S,\leq}]$ is a right $[[R^{S,\leq}]]$-module. For example, if $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the usual order, then $[M^{\mathbb{N}\cup\{0\},\leq}] \cong M[x^{-1}]$, the usual left $R[[x]]$-module introduced in [10] and [11], which is called the Macaulay-Northcott module in [12] and [13]. Further examples of modules of generalized inverse polynomials are given in [7].

Next, we explain some notations and facts involved. To any $r \in R$, we define $c_r \in [[R^{S,\leq}]]$ by

$$c_r(x) = \begin{cases} r, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0. \end{cases}$$

To any $m \in M$ and any $s \in S$, we associate the map $d_m^s \in [[M^{S,\leq}]]$ via

$$d_m^s(x) = \begin{cases} m, & \text{if } x = s, \\ 0, & \text{if } x \neq s. \end{cases}$$

For any $m \in M$ and any $s \in S$, we define $\phi_{s,m} \in [M^{S,\leq}]$ via

$$\phi_{s,m}(x) = \begin{cases} m, & \text{if } x = s, \\ 0, & \text{if } x \neq s. \end{cases}$$

Throughout this paper, R denotes a commutative ring with identity, all modules are unitary and $(S, \leq)$ is a strictly totally ordered monoid. In this situation, for every $0 \neq \phi \in [[M^{S,\leq}]]$ (resp. $0 \neq f \in [[R^{S,\leq}]]$), supp$(\phi)$ (resp. supp(f)) has a minimal element, we denote it by $\pi(\phi)$ (resp. $\pi(f)$). If $(S, \leq)$ satisfies the condition that $0 \leq s$ for all $s \in S$, then $(f\phi)(0) = f(0)\phi(0)$ for any $\phi \in [[M^{S,\leq}]]$ and any $f \in [[R^{S,\leq}]]$. If $(S, \leq)$ is also artinian, then by [7], $0 \leq s$ for any $s \in S$, and for any $0 \neq \varphi \in [M^{S,\leq}]$, supp$(\varphi)$ has a maximal element, we denote by $\sigma(\varphi)$.

3 Main results and proofs

Definition 3.1 *Let R be a commutative ring. An R-module M is said to be semi Hopfian (resp. semi co-Hopfian) if for any $r \in R$, the endomorphism of M induced by multiplication by r is an isomorphism, provided it is surjective (resp. injective).*

In [2, Theorem 3.4], it was shown that M is semi Hopfian R-module if and only if $M[X_1, \dots, X_n]$ is semi Hopfian $R[X_1, \dots, X_n]$-module if and only if $M[[X_1, \dots, X_n]]$ is semi Hopfian $R[[X_1, \dots, X_n]]$-module. Here we have:

Theorem 3.2 *Let $(S, \leq)$ be a strictly totally ordered monoid and satisfies the condition that $0 \leq s$ for any $s \in S$, R a commutative ring and M an R-module. Then $[[M^{S,\leq}]]$ is a semi Hopfian $[[R^{S,\leq}]]$-module if and only if M is a semi Hopfian R-module.*

Proof $\Longrightarrow$) Let $\alpha : M \longrightarrow M$ defined by $\alpha(m) = rm$ for some fixed $r \in R$ be a surjective R-homomorphism. Define $\beta : [[M^{S,\leq}]] \longrightarrow [[M^{S,\leq}]]$ via $\beta(\varphi) = c_r\varphi$ for any $\varphi \in [[M^{S,\leq}]]$. Then it is easy to see that β is an $[[R^{S,\leq}]]$-homomorphism. For any $\psi \in [[M^{S,\leq}]]$ and any $s \in S$, there exists an element $m_s \in M$ such that $\alpha(m_s) = \psi(s)$ since α is surjective. Define $\varphi : S \longrightarrow M$ via

$$\varphi(s) = \begin{cases} m_s, & \text{if } s \in \text{supp}(\psi), \\ 0, & \text{if } s \notin \text{supp}(\psi). \end{cases}$$

Clearly, supp(φ)=supp(ψ), and so $\varphi \in [[M^{S,\leq}]]$. For any $s \in S$,

$$\beta(\varphi)(s) = (c_r\varphi)(s) = r\varphi(s) = \begin{cases} rm_s, & s \in \text{supp}(\psi), \\ 0, & s \notin \text{supp}(\psi), \end{cases} = \psi(s).$$

This means that $\beta(\varphi) = \psi$, and thus β is a surjective $[[R^{S,\leq}]]$-homomorphism. Hence β is an isomorphism since $[[M^{S,\leq}]]$ is a semi Hopfian $[[R^{S,\leq}]]$-module. Let $m \in M$ be such that $rm = 0$. Then $\beta\left(d_m^0\right) = c_r d_m^0 = 0$. Thus $d_m^0 = 0$ and so $m = 0$ since β is an isomorphism. This means that α is an isomorphism.

$\Longleftarrow$) Let $\alpha : [[M^{S,\leq}]] \longrightarrow [[M^{S,\leq}]]$ defined by $\alpha(\varphi) = f\varphi$ for some fixed $f \in [[R^{S,\leq}]]$ be a surjective $[[R^{S,\leq}]]$-homomorphism. Define $\beta : M \longrightarrow M$ via $\beta(m) = f(0)m$ for any $m \in M$. Then it is easy to see that β is an R-homomorphism. For any $m \in M$, there exists an element $\varphi \in [[M^{S,\leq}]]$ such that $\alpha(\varphi) = d_m^0$ since α is surjective. Then $m = d_m^0(0) = \alpha(\varphi)(0) = (f\varphi)(0) = f(0)\varphi(0) = \beta(\varphi(0))$ since $0 \leq s$ for any $s \in S$. This implies that β is a surjective R-homomorphism, which must be an isomorphism since M is semi-Hopfian.

Let $\varphi \in [[M^{S,\leq}]]$ be such that $f\varphi = \alpha(\varphi) = 0$. Then $\beta(\varphi(0)) = f(0)\varphi(0) = (f\varphi)(0) = 0$, and so $\varphi(0) = 0$ since β is an isomorphism. Now, suppose that

$u \in S$ and for any $v \in S$ with $v < u$, $\varphi(v) = 0$. We will show that $\varphi(u) = 0$. Note that $\pi\left(\varphi - d^u_{\varphi(u)}\right) > u$. So $\pi\left(f\left(\varphi - d^u_{\varphi(u)}\right)\right) \geq \pi(f) + \pi\left(\varphi - d^u_{\varphi(u)}\right) > \pi(f) + u \geq u$. Thus

$$\begin{aligned}\beta(\varphi(u)) &= f(0)\varphi(u) = \left(fd^u_{\varphi(u)}\right)(u) = \left(fd^u_{\varphi(u)}\right)(u) + \left(f\left(\varphi - d^u_{\varphi(u)}\right)\right)(u)\\ &= (f\varphi)(u) = \alpha(\varphi)(u) = 0.\end{aligned}$$

Hence $\varphi(u) = 0$ since β is an isomorphism. Therefore $\varphi = 0$ and thus α is an isomorphism. □

Corollary 3.3 *Let $(S, \leq)$ be a strictly totally ordered monoid and satisfies the condition that $0 \leq s$ for any $s \in S$, R be a commutative ring and M, N be R-modules. Then the following are equivalent:*

(1) Any $[[R^{S,\leq}]]$-epimorphism $[[M^{S,\leq}]] \longrightarrow [[N^{S,\leq}]]$ induced by multiplication by some $f \in [[R^{S,\leq}]]$ is an isomorphism.

(2) Any R-epimorphism $M \longrightarrow N$ induced by multiplication by some $r \in R$ is an isomorphism.

Proof It is similar to the proof of Theorem 3.2. □

In [2, Theorem 3.5], it was shown that M is semi co-Hopfian R-module if and only if $M[X_1^{-1}, \ldots, X_n^{-1}]$ is semi co-Hopfian $R[X_1, \ldots, X_n]$-module if and only if $M[X_1^{-1}, \ldots, X_n^{-1}]$ is semi co-Hopfian $R[[X_1, \ldots, X_n]]$-module. Here we have:

Theorem 3.4 *Let $(S, \leq)$ be a strictly totally ordered monoid which is also artinian, R a commutative ring and M an R-module. Then $[M^{S,\leq}]$ is a semi co-Hopfian $[[R^{S,\leq}]]$-module if and only if M is a semi co-Hopfian R-module.*

Proof $\Longrightarrow$) Let $\alpha : M \longrightarrow M$ defined by $\alpha(m) = rm$ for some fixed $r \in R$ be an injective R-homomorphism. Define $\beta : [M^{S,\leq}] \longrightarrow [M^{S,\leq}]$ via $\beta(\varphi) = c_r\varphi$ for any $\varphi \in [M^{S,\leq}]$. Then it is easy to see that β is an $[[R^{S,\leq}]]$-homomorphism. If $\beta(\varphi) = 0$ where $\varphi \in [M^{S,\leq}]$. Then, for any $s \in S$,

$$0 = \beta(\varphi)(s) = (c_r\varphi)(s) = \sum_{x \in S} c_r(x)\varphi(x + s) = r\varphi(s) = \alpha(\varphi(s)).$$

Thus $\varphi(s) = 0$ since α is injective. Hence $\varphi = 0$. This means that β is injective which must be an isomorphism since $[M^{S,\leq}]$ is semi co-Hopfian. Now, for any $m \in M$, there exists a $\varphi \in [M^{S,\leq}]$ such that $c_r\varphi = \beta(\varphi) = \phi_{0,m}$. Thus

$$m = \phi_{0,m}(0) = (c_r\varphi)(0) = \sum_{x \in S} c_r(x)\varphi(x) = r\varphi(0) = \alpha(\varphi(0)).$$

This means that α is surjective.

$\Longleftarrow$) Let $\alpha : [M^{S,\leq}] \longrightarrow [M^{S,\leq}]$ defined by $\alpha(\varphi) = f\varphi$ for some fixed $f \in [[R^{S,\leq}]]$ be an injective $[[R^{S,\leq}]]$-homomorphism. Define $\beta : M \longrightarrow M$ via $\beta(m) = f(0)m$ for any $m \in M$. Then it is easy to see that β is an R-homomorphism. If $\beta(m) = 0$, then for any $s \in S$,

$$\alpha(\phi_{0,m})(s) = (f\phi_{0,m})(s) = \sum_{x\in S} f(x)\phi_{0,m}(x+s) = \begin{cases} f(0)m = \beta(m), & s = 0, \\ 0, & s > 0, \end{cases} = 0,$$

which implies that $\alpha(\phi_{0,m}) = 0$, and so $\phi_{0,m} = 0$ since α is injective. Thus $m = 0$. This means that β is injective which must be an isomorphism since M is semi co-Hopfian. Now we show that α is surjective.

Let $\varphi \in [M^{S,\leq}]$ with $\sigma(\varphi) = s$. If $s = 0$, set $m = \beta^{-1}(\varphi(0))$. Then, for any $t \in S$,

$$\alpha(\phi_{0,m})(t) = \begin{cases} f(0)m = \beta(m) = \beta\left(\beta^{-1}(\varphi(0))\right) = \varphi(0), & t = 0, \\ 0, & t > 0, \end{cases} = \varphi(t),$$

which means that $\alpha(\phi_{0,m}) = \varphi$.

Now, suppose that $0 < s$. Assume that for any $\psi \in [M^{S,\leq}]$ with $\sigma(\psi) < s$, there exists $\psi' \in [M^{S,\leq}]$ such that $\alpha(\psi') = \psi$. Since $0 \neq \varphi(s) \in M$, there exists an $m \in M$ such that $\varphi(s) = \beta(m) = f(0)m$. For any $s \leq t \in S$, from

$$\alpha(\phi_{s,m})(t) = (f\phi_{s,m})(t) = \sum_{x\in S} f(x)\phi_{s,m}(x+t) = \begin{cases} f(0)m = \varphi(s), & t = s, \\ 0, & s < t, \end{cases} = \varphi(t)$$

it follows that $\sigma(\varphi - \alpha(\phi_{s,m})) < s$. By the hypothesis, there exists $\varphi' \in [M^{S,\leq}]$ such that $\varphi - \alpha(\phi_{s,m}) = \alpha(\varphi')$. Thus $\varphi = \alpha(\varphi' + \phi_{s,m})$.

Therefore, by the transfinite induction, we have shown that α is surjective.
□

Corollary 3.5 *Let $(S, \leq)$ be a strictly totally ordered monoid which is also artinian, R be a commutative ring and M, N be R-modules. Then the following are equivalent:*

(1) Any $[[R^{S,\leq}]]$-monomorphism $[M^{S,\leq}] \longrightarrow [N^{S,\leq}]$ induced by multiplication by some $f \in [[R^{S,\leq}]]$ is an isomorphism.

(2) Any R-monomorphism $M \longrightarrow N$ induced by multiplication by some $r \in R$ is an isomorphism.

Proof It is similar to the proof of Theorem 3.4. □

4 Corollaries

Corollary 4.1 *Let S be a torsion-free and cancellative monoid, $(S, \leq)$ be artinian and narrow, R a commutative ring and M an R-module. Then*

(1) $[[M^{S,\leq}]]$ *is a semi Hopfian* $[[R^{S,\leq}]]$*-module if and only if* M *is a semi Hopfian* R*-module.*

(2) $[M^{S,\leq}]$ *is a semi co-Hopfian* $[[R^{S,\leq}]]$*-module if and only if* M *is a semi co-Hopfian* R*-module.*

Proof If $(S,\leq)$ is torsion-free and cancellative, then by [14, 3.3], there exists a compatible strict total order $\leq'$ on S, which is finer than $\leq$, that is, for any $s,t\in S, s\leq t$ implies $s\leq' t$. Since $(S,\leq)$ is artinian and narrow, by [14, 2.5] it follows that $(S,\leq')$ is artinian and narrow. Thus, by Theorem 3.2, $[[M^{S,\leq'}]]$ is a semi Hopfian $[[R^{S,\leq'}]]$-module if and only if M is a semi Hopfian R-module, and by Theorem 3.4, $[M^{S,\leq'}]$ is a semi co-Hopfian $[[R^{S,\leq'}]]$-module if and only if M is a semi co-Hopfian R-module. On the other hand, since $(S,\leq)$ is narrow, by [14, 4.4], $[[R^{S,\leq}]]=[[R^{S,\leq'}]]$. Clearly $[[M^{S,\leq}]]=[[M^{S,\leq'}]]$ and $[M^{S,\leq}]=[M^{S,\leq'}]$. Now the result follows. □

Any submonoid of the additive monoid $\mathbb{N}\cup\{0\}$ is called a *numerical* monoid. We have

Corollary 4.2 *Let* S *be a numerical monoid and* $\leq$ *the usual natural order of* $\mathbb{N}\cup\{0\}$*,* R *a commutative ring and* M *an* R*-module. Then*

(1) $[[M^{S,\leq}]]$ *is a semi Hopfian* $[[R^{S,\leq}]]$*-module if and only if* M *is a semi Hopfian* R*-module.*

(2) $[M^{S,\leq}]$ *is a semi co-Hopfian* $[[R^{S,\leq}]]$*-module if and only if* M *is a semi co-Hopfian* R*-module.*

If S is the multiplicative monoid $(\mathbb{N},\cdot)$, endowed with the usual order $\leq$, then $[[R^{(\mathbb{N},\cdot),\leq}]]$ is the ring of arithmetical functions with values in R, endowed with the Dirichlet convolution:

$$(fg)(n)=\sum_{d|n}f(d)g(n/d),\quad \text{for each } n\geq 1.$$

If M is a left R-module, then $[[M^{(\mathbb{N},\cdot),\leq}]]$ is a left $[[R^{(\mathbb{N},\cdot),\leq}]]$-module with scalar multiplication:

$$(f\phi)(n)=\sum_{d|n}f(d)\phi(n/d),\quad \text{for each } n\geq 1,$$

where $f\in[[R^{(\mathbb{N},\cdot),\leq}]]$ and $\phi\in[[M^{(\mathbb{N},\cdot),\leq}]]$. The left $[[R^{(\mathbb{N},\cdot),\leq}]]$-module $[M^{(\mathbb{N},\cdot),\leq}]$ is the set $\left\{\sum_{i=1}^{n}m_ix^{-i}\mid m_i\in M, i=1,2,\dots,n,n\in\mathbb{N}\right\}$ with scalar multiplication as below:

$$\left(\sum_{i\geq 1}r_ix^i\right)\left(\sum_{j\geq 1}m_jx^{-j}\right)=\sum_{j\geq 1}\left(\sum_{i\geq 1}r_im_{i\cdot j}\right)x^{-j}$$

where $\sum_{i\geq 1}r_ix^i\in[[R^{(\mathbb{N},\cdot),\leq}]]$ and $\sum_{j\geq 1}m_jx^{-j}\in[M^{(\mathbb{N},\cdot),\leq}]$.

Corollary 4.3 *Let R be a commutative ring and M an R-module. Then*

(1) $[[M^{(\mathbb{N},\cdot),\leq}]]$ is a semi Hopfian $[[R^{(\mathbb{N},\cdot),\leq}]]$-module if and only if M is a semi Hopfian R-module.

(2) $[M^{(\mathbb{N},\cdot),\leq}]$ is a semi co-Hopfian $[[R^{(\mathbb{N},\cdot),\leq}]]$-module if and only if M is a semi co-Hopfian R-module.

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