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Some Remarks on Ideals of an Incline

A. R. Meenakshi and N. Jeyabalan

Department of Mathematics, Karpagam University
Coimbatore – 641 021, India
arm_meenakshi@yahoo.co.in, balan.maths84@gmail.com

Abstract

In this paper we have discussed the relations between Maximal ideal, Prime ideal and irreducible ideal in an incline R.

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1. Introduction

Inclines are additively idempotent semirings in which products are less than (or) equal to either factor. The concept of incline was introduced by Cao and later it was developed by Cao, et.al, in [2]. Recently a survey on incline was made by Kim and Roush [4]. Incline algebra is a generalization of both Boolean and fuzzy algebras and it is a special type of a semiring. It has both a semiring structure and a poset structure. It can also be used to represent automata and other mathematical systems, to study inequalities for non-negative matrices of polynomials.

Ahn, Jun and Kim have proved that "In an incline R, every prime ideal of R is equivalent to irreducible ideal of R" which forms a part of Theorem 3.18 of [1]. Here we have proved that, in an incline R, every prime ideal of R is an irreducible ideal of R and illustrated with an example that the converse need not be true. In section 2, we present the basic definitions and required results of an incline R. In section 3, we have proved that in an incline R, every Maximal ideal is an irreducible ideal and Prime ideal is an irreducible ideal and illustrated with the suitable examples for the converse need not be true. And thereby disproving Theorem 3.18 of [1]. We have illustrated with suitable examples that there is no relation between Maximal ideal and Prime ideal in an incline.

2. Preliminaries

In this section, we present some definitions and required results on an incline.

Definition 2.1

An *incline* is a non-empty set R with binary operations addition and multiplication denoted as +, $\cdot$ defined on R x R $\rightarrow$ R such that for all x, y, z $\in$ R,

$x+y = y+x$, $x+(y+z) = (x+y)+z$,

$x(y+z) = xy+xz$, $(y+z)x = yx+zx$,

$x(yz) = (xy)z$, $x+xy = x$,

$x+x = x$, $y+xy = y$.

An incline R is said to be *commutative* if xy = yx, for all x, y $\in$R.

In [2], Authors have considered the incline as commutative incline.

Definition 2.2

(R, $\leq$) is an incline with order relation '$\leq$' defined as, $x \leq y$ if and only if $x+y = y$, for x, y $\in$ R. If $x \leq y$ then y is said to *dominate* x.

Property 2.3

For x, y in an incline R, $x+y \geq x$ and $x+y \geq y$.

Property 2.4

For x, y in an incline R, $xy \leq x$ and $xy \leq y$.

Definition 2.5

A *subincline* of an incline R is a non–empty subset I of R which is closed under the incline operations addition and multiplication.

Definition 2.6

A subincline I is said to be an *ideal* of an incline R if x $\in$ I and $y \leq x$ then y $\in$ I.

We call this ideal as "ideal as a subincline".

Definition 2.7

A proper ideal P of an incline R is said to be *prime* if for all x, y∈R, xy∈P implies either x∈P (or) y∈P.

Definition 2.8

An ideal M of an incline R is said to be *maximal* if M ≠ R and for every ideal N with M ⊆ N ⊆ R, either N = M (or) N = R.

Definition 2.9

A proper ideal I of an incline R is said to be *irreducible* if I = A ∩ B implies I = A (or) I = B, for some ideals A, B of R.

3. Some remarks on ideals of an incline

In this section, we have proved that, in an incline R, every Maximal ideal is an irreducible ideal and Prime ideal is an irreducible ideal. Illustrated with the suitable examples that the converse is not true and there is no relation between Maximal ideal and Prime ideal. Hence, Theorem 3.18 of [1] fails for an incline.

Theorem 3.1

In an incline R, every maximal ideal M of R is an irreducible ideal.

Proof

Let M be a maximal ideal of R. Suppose M is not irreducible, then M = A ∩ B, for any two ideals A and B of R implies M ≠ A and M ≠ B.
Thus, M ⊂ A ⊆ R and M ⊂ B ⊆ R implies M ⊂ A ∩ B.
Which is a contradiction to our hypothesis.
Hence M is an irreducible ideal.
□

Remark 3.2

The converse of the Theorem (3.1) is not true.This is illustrated in the following example:

Example 3.3

Let us consider R = {[0, 1], +, .} a commutative incline, where x + y = max{x, y} and x .y = xy (usual multiplication). Here, all the ideals of R are the closed intervals of the form [0, a], for some a ∈ R.

For, A = [0, a] and B = [0, b] if I = A ∩ B, since a, b ∈ [0, 1] are comparable, we have either I = A (or) I = B.
Thus, all the ideals are irreducible ideals.
For, instance, let I = [0, .2] be an ideal of R, which is irreducible. But I is not maximal. Since, for the ideal [0, .3] of R, we have [0, .2] ⊆ [0, .3] ⊆ R.

Theorem 3.4

In an incline R, every Prime ideal of R is an irreducible ideal.

Proof

Let P be a prime ideal of R. To show that, P is an irreducible ideal, whenever P = A ∩ B, for some ideals A, B of R, it is enough to show that either P = A (or) P = B.
For, Let x ∈ A and y ∈ B, be arbitrary elements of A and B.
Then by incline property (2.4), we have xy ≤ x and xy ≤ y. Since A and B are ideals of R, by Definition (2.6), xy ∈ A and xy ∈ B implies xy ∈ A ∩ B = P.
Since P is a prime ideal of R, by Definition (2.7) either x ∈ P (or) y ∈ P.
That is, either A = P (or) B = P.
Thus, P is an irreducible ideal.
□

Remark 3.5

The converse of the Theorem (3.4) is not true. This is illustrated in the following example:

Example 3.6

Let us consider the incline R in Example (3.3) in which all the ideals are irreducible ideals, hence R is an irreducible incline.
Let I = [0, .2] be an ideal of R.
Let us consider the elements (.3) and (.4) ∈ R, then (.3) × (.4) = (.12) ∈ I, but (.3) ∉ I and (.4) ∉ I. Therefore, by Definition (2.7) I is not prime. Thus Theorem 3.18 of [1] fails.

Remark 3.7

In the proof of Theorem 3.18 of [1], the authors have assumed that $I_1 = I \cup \{x\}$ and $I_2 = I \cup \{y\}$ are ideals of R, for an ideal I of R and x, y ∈ R. We have observed that in general whenever I is an ideal of R and by adjoining an element to I need not be an ideal. This is illustrated in the following example:

Example 3.8

Let us consider the incline R in Example (3.3) and I = [0, .2] is an ideal of R. Consider, (.4), (.5) $\in$ R then (.4) $\times$ (.5) = (.2) $\in$ I.

Here, I_1 = [0, .2] $\cup$ {.4} and I_2 = [0, .2] $\cup$ {.5} are not ideals. Since, by Definition (2.6),

(.3) $\leq$ (.4) for some (.3) $\in$ R but (.3) $\notin I_1$. Similarly, (.4) $\leq$ (.5) for some (.4) $\in$ R but (.4) $\notin I_2$.

Thus, I_1 and I_2 are not ideals.

However I = $I_1 \cap I_2$ is an ideal of R.

Remark 3.9

It is well known that in a Commutative Ring with unit element "Every Maximal ideal is a Prime ideal" (p.167 of [3]). However this fails for an incline, a special type of a semiring. That is, there is no relation between Maximal ideal and Prime ideal in an incline. This is illustrated in the following examples:

Example 3.10

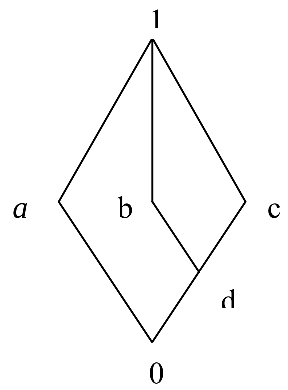
Let us consider R = {[0, 1], +, .} a commutative, regular incline, where x+y = max{x, y} and xy = min{x, y}.

Here all the ideals of R are the closed intervals of the form [0, a] for some a$\in$R and all the ideals are prime ideals.

For instance, let I = [0, .3] be an ideal of R which is a prime. Since for the ideal [0, .4] of R we have [0, .3] $\subset$ [0, .4] $\subset$ R, I is not maximal.

Example 3.11

Let R = {0,a,b,c,d,1} be an incline. Define •: R•R$\rightarrow$R by x•y = d for all x,y$\in${b,c,d,1} and 0 otherwise. For this incline R, ϑ(R), the set of all ideals of R = {I_1, I_2, I_3, I_4, I_5}, where I_1={0, a}, I_2={0, d}, I_3={0, b, d}, I_4={0, c, d} and I_5 = R



Here $I_3 = \{0, b, d\}$ is an ideal of R which is a maximal ideal.
We have $a.c = 0 \in I$ but $a \notin I$ and $c \notin I$. Thus I is not a prime ideal.

References

[1] S.S. Ahn, Y.B. Jun, H.S. Kim, Ideals and Quotients of Incline Algebras, Comm. Korean Math. Soc. 16 (2001), 2573 – 583.

[2] Z.Q. Cao, K.H. Kim, F.W. Roush, Incline Algebra and Applications, John Wiley and sons, New York, 1984.

[3] I.N. Herstein, Topics in Algebra, John Wiley and Sons(Asia) Pte Ltd, Singapore, 2000.

[4] K.H. Kim, F.W. Roush, Inclines and Incline Matrices: a survey, Linear Algebra Appl. 379 (2004), 457 – 473.

[5] K. H. Kim and F. W. Roush, Inclines of algebraic structures, Fuzzy Sets and Systems 72 (1995), 189 – 196.

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