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On T-Rough Prime and Primary Submodules

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Abstract

Roughness in modules have been investigated by B. Davvaz and M. Mahdavipour in 2006 [5]. The purpose of this paper is to introduce and discuss the concept of T-rough prime and primary submodules which is a generalization the lower and upper approximation submodules over a commutative ring. We define a set-valued homomorphism on a module and study some properties of it . We prove some results for decomposable modules.

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1 Introduction

The notion of rough sets has been introduced by Z. Pawlak in his papers [13, 14], Z. Pawlak and A. Skowron [15] and T. Iwinski [10]. It soon invoked a natural question concerning possible connection between rough sets and algebraic systems. The algebraic approach to rough sets have been given and studied by Bonikowaski [2]. Biswas and Nanda [1] introduced the notion of rough subgroups. Kuroki [12] introduced the notion of rough ideals in a semigroups. Qi-Mei Xiao, Zhen-Liang [16] studied rough prime ideals and rough fuzzy prime ideals in semigroups. Davvaz [4] introduced the notion of rough subring with respect to an ideal of a ring. Davvaz and Mahdavipour [5] discussed roughness in modules. Osman Kazanci, Davvaz [11] discussed the structure on rough prime (primary) ideals. In [10, 18] were considered some other results. Davvaz [3] introduced T- rough sets and set-valued homomorphism on a group. In [17], S. Yamak, O. Kazanci, B. Davvaz introduced the generalized lower and upper approximation in a ring. S. B. Hosseini et al.[6, 7, 8] introduced and discussed T-rough sets in modules, semigroups and rings. Rough sets are a suitable mathematical model of vague concepts, i.e., concepts without sharp boundaries. One of the things which distinguishes the modern approach to commutative algebra is the greater emphasis on modules, rather than just on ideals.

In this paper is discussed and introduced to the set-valued homomorphism on a module, the concept T-rough prime and primary submodule of a module over a commutative ring and is proved some interesting properties.

2 Preliminary Notes

The following definitions and preliminaries are required in the sequel of our work and hence presented in brief. Suppose that U is a nonempty set. A partition or classification of U is a family Θ of nonempty subsets of U such that each element of U is contained in exactly one element of Θ. Recall that an equivalence relation θ on a set U is a reflexive, symmetric and transitive binary relation on U. Each partition induces an equivalence relation on U. If θ is an equivalence relation on U, then for every $x \in U, [x]_\theta$ denotes the equivalence class of θ determined by x.

Definition 2.1. *[13, 14] A pair (U, θ) where $U \neq \emptyset$ and θ is an equivalence relation on U is called an approximation space.*

Definition 2.2. *[13, 14] For an approximation space (U, θ) by a rough approximation in (U, θ) we mean a mapping $Apr : P(U) \longrightarrow P(U) \times P(U)$ defined by for every $X \in P(U), Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$, where*

$$\underline{Apr}(X) = \{x \in U \mid [x]_\theta \subseteq X\} \ ; \ \overline{Apr}(X) = \{x \in U \mid [x]_\theta \cap X \neq \emptyset\}.$$

$\underline{Apr}(X)$ is called the lower rough approximation of X in (U, θ) whereas $\overline{Apr}(X)$ is called the upper rough approximation of X in (U, θ).

Definition 2.3. *[13, 14] Given an approximation space (U, θ) a pair (A, B) in $P(U) \times P(U)$ is called a rough set in (U, θ) if $(A, B) = (\underline{Apr}(X), \overline{Apr}(X))$ for some $X \in P(U)$.*

Proposition 2.4. *[13, 14] Let U be a nonempty set and θ an equivalence relation on U. For every subsets $A, B \subseteq U$, we have*

(i) $\underline{Apr}(A) \subseteq A \subseteq \overline{Apr}(A)$;

(ii) *If $A \subseteq B$, then $\underline{Apr}(A) \subseteq \underline{Apr}(B)$ and $\overline{Apr}(A) \subseteq \overline{Apr}(B)$;*

(iii) $\underline{Apr}(A \cap B) = \underline{Apr}(A) \cap \underline{Apr}(B)$;

(iv) $\overline{Apr}(A) \cup \overline{Apr}(B) = \overline{Apr}(A \cup B)$.

3 Set-valued Homomorphism and T-rough submodule

In this section, we define the concept of a set-valued homomorphism. We show that every module homomorphism is a set-valued homomorphism. We generalize the rough submodule that is called T-rough submodule.
Throughout in this section and next one, $P^*(Y)$ is denoted by the set of all nonempty subsets of Y and R is a commutative ring with identity and all R-modules are unitary.

Definition 3.1. [3] *Let X and Y be two nonempty sets and $\emptyset \neq B \subseteq Y$. Let $T : X \to P^*(Y)$ be a set-valued mapping. The lower inverse and upper inverse of B under T are defined by*

$$L_T(B) = \{x \in X \mid T(x) \subseteq B\} \ ; \ U_T(B) = \{x \in X \mid T(x) \cap B \neq \emptyset\}$$

, respectively.

Definition 3.2. [3] *Let $X$and Y be two nonempty sets and $B \in P^*(Y)$. Let $T : X \to P^*(Y)$ be a set-valued mapping. $(L_T(B), U_T(B))$ is called a T- rough set with respect to B.*

Proposition 3.3. [9] *Let X and Y be two nonempty sets and $A, B \in P^*(Y)$. Let $T : X \to P^*(Y)$ be a set-valued mapping, then the following hold:*

(i) $L_T(Y) = X = U_T(Y)$ *and* $L_T(A) \subseteq U_T(A)$;

(ii) $L_T(A \cap B) = L_T(A) \cap L_T(B)$ *and* $U_T(A \cup B) = U_T(A) \cup U_T(B)$;

(iii) $A \subseteq B$ *implies* $L_T(A) \subseteq L_T(B)$ *and* $U_T(A) \subseteq U_T(B)$;

(iv) $L_T(A) \cup L_T(B) \subseteq L_T(A \cup B)$ *and* $U_T(A \cap B) \subseteq U_T(A) \cap U_T(B)$.

Example 3.4. *Let (U, θ) be an approximation space and $T : U \to P^*(U)$ be a set-valued mapping where $T(x) = [x]_\theta$, then for any $B \subseteq U$, $L_T(B) = \underline{Apr}(B)$ and $U_T(B) = \overline{Apr}(B)$. So rough sets are T-rough sets. In fact, T-rough sets are a generalization of rough sets.*

Definition 3.5. *Let M be an R-module. If A be a submodule of M and $\emptyset \neq S \subseteq R$, then SA denotes the set of $\{\sum_{i=1}^{n} s_i a_i \mid s_i \in S, a_i \in A, n \in \mathbf{N}\}$.*

Definition 3.6. *Let M and N be R-modules and $T : M \to P^*(N)$ be a set-valued mapping. T is called a set-valued homomorphism if*

(*i*) $T(m_1 + m_2) = T(m_1) + T(m_2)$;

(*ii*) $T(rm) = rT(m)$.

for all $r \in R$ *and* $m, m_1, m_2 \in M$.

It is clear that $T(0) = \{0\}$ and $T(-m) = -T(m)$ for all $m \in M$.

Example 3.7. *Let* $f : M \to N$ *be an* R*-module homomorphism and* $T : M \to P^*(N)$ *be a set valued mapping where* $T(m) = \{f(m)\}$ *for all* $m \in M$. *Then* T *is a set-valued homomorphism. So, a module homomorphism induce a set-valued homomorphism.*

The following theorem and lemma have been proved in [9].

Theorem 3.8. *Let* $T : M \to P^*(N)$ *be a set-valued homomorphism. If* A *be a submodule of* N *and* S *is a nonempty subset of* R. *Then*

(i) $SL_T(A) \subseteq L_T(SA)$;

(ii) $SU_T(A) \subseteq U_T(SA)$.

Lemma 3.9. *Let* A *be a submodule of* N *and* $T : M \to P^*(N)$ *be a set-valued homomorphism, then* $L_T(B)$ *and* $U_T(B)$ *are submodules of* M.

Definition 3.10. *If* $L_T(B)$ *and* $U_T(B)$ *be submodules of* M *for some* $B \in P^*(N)$, *then* $(L_T(B), U_T(B))$ *is called a* T*-rough submodule of* M *with respect to* B.

Corollary 3.11. *Suppose* $T : M \to P^*(N)$ *be a set-valued homomorphism and* B *be an submodule of* N, *then* $(L_T(B), U_T(B))$ *is a* T*-rough submodule of* M *with respect to* B.

4 T-rough prime submodules

In this section we define the T-rough prime submodule. Next, check that when the T-lower and the T-upper of any submodule of N are prime submodules of M. First, we require some classical definitions.

Definition 4.1. *(i) An ideal* P *is called a prime ideal if* $P \neq R$ *and if* $a, b \in R$ *and* $ab \in P$, *then* $a \in P$ *or* $b \in P$.

(ii) Let N *be a submodule of* M. *Define* $(N : M) = \{r \in R | rM \subseteq N\}$.

(iii) Let M *be an* R*-module. A proper submodule* N *of* M *is called prime submodule if given* $r \in R$, $m \in M$, $rm \in N$ *implies that* $m \in N$ *or* $r \in (N : M)$.

(iv) An ideal I of R is called a maximal ideal, if $I \neq R$ and J be any ideal of R such that $I \subseteq J$, then $I = J$ or $J = R$.
The maximal submodule of a module is defined similarly.

(v) Annihilator R-module M is the set $\{r \in R \mid rM = 0\}$ and is denoted by $AnnM$. It is clear that $AnnM$ is an ideal of R.

(vi) An R-module M is called torsion-free if $r \in R$, $m \in M$ and $rm = 0$ implies that $r = 0$ or $m = 0$.

Example 4.2. *A ring R itself is an R-module. Every prime ideal of R is a prime submodule and the maximal ideal is a maximal submodule of R.*

Theorem 4.3. *Let M, N be R-modules and A be a prime submodule of N and $T : M \to P^*(N)$ be a set-valued homomorphism. Then $U_T(A)$ and $L_T(A)$ are equal M or prime submodules of M.*

Proof. Suppose that $U_T(A) \neq M$ and $r \in R$, $m \in M$ such that $rm \in U_T(A)$ and $m \notin U_T(A)$. Thus $T(m) \cap A = \varnothing$. But $T(rm) \cap A \neq \varnothing$. Since T is a homomorphism, $rT(m) \cap A \neq \varnothing$. Hence there exists $n \in T(m)$ such that $rn \in A$. Since A is a prime submodule of N and $T(m) \cap A = \varnothing$, $rN \subseteq A$. Whence $U_T(rN) \subseteq U_T(A)$. By Theorem 3.8, $rU_T(N) \subseteq U_T(rN)$. By Proposition 3.3(i), $U_T(N) = M$. Consequently, $rM \subseteq U_T(A)$ (i.e. $r \in (U_T(A) : M)$).
Now show that if $L_T(A) \neq M$, then $L_T(A)$ is a prime submodule. Assume that $r \in R$, $m \in M$ such that $rm \in L_T(A)$ and $m \notin L_T(A)$ which implies $T(m) \nsubseteq A$. But $T(rm) \subseteq A$. Since T is a set-valued homomorphism, $rT(m) \subseteq A$. So there exists $n \in T(M)$ such that $n \notin A$ and $rn \in A$. Since A is a prime submodule of N, $rN \subseteq A$. Thus $L_T(rN) \subseteq L_T(A)$. By Theorem 3.8, $rL_T(N) \subseteq L_T(A)$. By Proposition 3.3(i), $L_T(N) = M$. Consequently, $rM \subseteq L_T(A)$ (i.e. $r \in (L_T(A) : M)$).

Lemma 4.4. *Let N be a submodule of M. Then $(N : M) = Ann(M/N)$ where $Ann(M/N)$ is the annihilator R-module M/N.*

Proof. We have

$$\begin{aligned} x \in Ann(M/N) &\Leftrightarrow x(M/N) = 0 \\ &\Leftrightarrow (xM + N)/N = 0 \\ &\Leftrightarrow xM + N = N \\ &\Leftrightarrow xM \subseteq N \Leftrightarrow x \in (N : M). \end{aligned}$$

Theorem 4.5. *Let M be an R-module. Then a submodule N of M is prime if and only if $P = (N : M)$ is a prime ideal of R and M/N is a torsion-free R/P-module.*

Proof. See in [6].

The proof of the following corollaries are similar to Corollary 2.21 in [6].

Corollary 4.6. *Let M, N be R-modules and A be a submodule of N and $T : M \to P^*(N)$ be a set-valued homomorphism. If $(U_T(A) : M)$ is a maximal ideal of R, then $U_T(A)$ is a prime submodule.*

Corollary 4.7. *Let M, N be R-modules and A be a submodule of N and $T : M \to P^*(N)$ be a set-valued homomorphism. If $(L_T(A) : M)$ is a maximal ideal of R, then $L_T(A)$ is a prime submodule.*

Lemma 4.8. *If N is a maximal submodule of M, then N is a prime submodule.*

Proof. The proof is similar to Corollary 2.22 in [6].

Corollary 4.9. *Let M, N be R-modules and A be a submodule of N and $T : M \to P^*(N)$ be a set-valued homomorphism. If $L_T(A)$ is a maximal submodule of M, then $L_T(A)$ is a prime submodule.*

Corollary 4.10. *Let M, N be R-modules and A be a submodule of N and $T : M \to P^*(N)$ be a set-valued homomorphism. If $U_T(A)$ is a maximal submodule of M, then $U_T(A)$ is a prime submodule.*

5 T-rough primary submodules

In this section, at first we discuss the primary submodule, then show that in many situation the lower and upper inverse of A under T are primary submodules.

Definition 5.1. *Let M be an R-module and N be a submodule of M. Then $\sqrt{N} = \{r \in R |\ r^k M \subseteq N$ for some positive integer $k\}$. $\sqrt{N}$ is called the radical of N.*

Definition 5.2. *Let N be proper submodule of M. Then we say that N is a primary submodule if $r \in R, m \in M$ and $rm \in N$ implies that $m \in N$ or $r \in \sqrt{N}$.*

Example 5.3. *Suppose that $M = \mathbb{Z}$. The primary submodules of M are 0 and $< p^n >$, where p is a prime number and $n \in \mathbb{N}$.*

Theorem 5.4. *Let M, N be two R-modules, A be a primary submodule of N and $T : M \to P^*(N)$ be a set-valued homomorphism. Then $L_T(A)$ and $U_T(A)$ are equal M or primary submodules of M.*

Proof. Suppose that $U_T(A) \neq M$, $r \in R$ and $m \in M$ such that $rm \in U_T(A)$ and $m \notin U_T(A)$. Clearly, $T(m) \cap A = \varnothing$. But $T(rm) \cap A \neq \varnothing$. Since T is a set-valued homomorphism, $rT(m) \cap A \neq \varnothing$. Hence there exists $n \in T(m)$ such that $rn \in A$. Since A is a primary submodule of N and $T(m) \cap A = \varnothing$, $r^k N \subseteq A$ for some positive integer k. So $r^k U_T(N) \subseteq U_T(A)$ by Proposition 3.3 (iii) and Theorem 3.3. Hence we have $r^k M \subseteq U_T(A)$ for some positive integer k. Therefore $r \in \sqrt{U_T(A)}$.
The proof for $L_T(A)$ is similarly.

Clearly, every prime submodule is primary submodule. Hence $U_T(A)$ and $L_T(A)$ in previous corollaries are primary submodules of M.

Lemma 5.5. *Let M be an R-module and N be a primary submodule of M. Then $\sqrt{N}$ is a prime ideal of R.*

Proof. Assume that $ab \in \sqrt{N}$ and $b \notin \sqrt{N}$. Hence $a^k b^k M \subseteq N$ for some positive integer k. Since N is a primary and $b \notin \sqrt{N}$ and $a^k \in \sqrt{N}$, $a^{ks} M \subseteq N$ for some positive integer s. Therefore $a \in \sqrt{N}$.

Definition 5.6. *If N be a primary submodule of M and $\sqrt{N} = P$, then N is said to be P-primary.*

Theorem 5.7. *Let M, N be two R-modules and $T : M \to P^*(N)$ be a set-valued homomorphism. If A is a P-primary submodule of N, then $L_T(A)$ and $U_T(A)$ are P-primary submodules of M.*

Proof. By Theorem 5.4, $L_T(A)$ and $U_T(A)$ are primary submodules. Now we show that $\sqrt{U_T(A)} = \sqrt{L_T(A)} = P$. Suppose that $r \in P$. Then $r^k N \subseteq A$ for some positive integer k. Hence $r^k U_T(N) \subseteq U_T(A)$. So $r^k M \subseteq U_T(A)$. That is $r \in \sqrt{U_T(A)}$. Therefore $P \subseteq \sqrt{U_T(A)}$. Similarly, suppose $r \in \sqrt{U_T(A)}$, then $r^k M \subseteq U_T(A)$ for some positive integer k. Since $U_T(A)$ is primary, there exists $m \in M$ such that $m \notin U_T(A)$. So $T(m) \cap A = \varnothing$. But $r^k T(m) \cap A \neq \varnothing$. Hence there exists $n \in T(m)$ such that $r^k n \in A$. Since A is primary, $r \in \sqrt{A}$ $= P$. Therefore $\sqrt{U_T(A)} = P$. Similarly, $\sqrt{L_T(A)} = P$.

Definition 5.8. *A primary decomposition of a submodule N of M is expression of N as a finite intersection of primary submodules, say $N = \cap_{i=1}^{n} K_i$.*

Definition 5.9. *We say that a submodule N of M is decomposable if it has a primary decomposition.*

The following example shows that the lower and upper inverse of A are not decomposable however A is decomposable.

Example 5.10. *Let $M = \mathbb{Z}_{12}$. Then M is a $\mathbb{Z}-$module. Suppose that $T : M \to P^*(M)$ be a set-valued homomorphism with $T(m) = \{0\}$ for all $m \in M$. Assume that $A = \{0, 2, 4, 6, 8, 10\}$, then $U_T(A) = \mathbb{Z}_{12} = L_T(A)$.*

Lemma 5.11. *Let M be an R-module and K be a decomposable submodule of M. Let $K = \cap_{i=1}^n K_i$ be a primary decomposition of K such that $\sqrt{K_i} = P$ $(1 \leq i \leq n)$. Then K is P-primary.*

Proof. Clearly, $\sqrt{K} = \sqrt{\cap_{i=1}^n K_i} = \cap_{i=1}^n \sqrt{K_i} = P$. Suppose that $r \in R, m \in M, rm \in K$ and $m \notin K$. Then $rm \in K_i$ and $m \notin K_i$ for some i. Since K_i is primary, $r \in P$.

We obtain the following corollary from Proposition 3.3, Lemma 5.7, and Lemma 5.11.

Corollary 5.12. *Let M, N be two R-modules and A be a decomposable submodule of N. Let $A = \cap_{i=1}^n K_i$ be a primary decomposition of A such that $\sqrt{K_i} = P$ $(1 \leq i \leq n)$ and $T : M \to P^*(N)$ be a set-valued homomorphism. Then $L_T(A)$ is equal M or is P-primary and decomposable.*

6 Conclusions

Modules owe their importance to the fact that so many models arising in the solutions of specific problems turn out to be modules. For this reason the basic concepts introduced here have exhibited some universality and are applicable in so many diverse contexts. These concepts are important and effective tools in algebra, linear algebra, vector spaces and physics. In this work, the lower T-rough and upper T-rough submodules are formulated in the context of module theory. We introduced the notion of the generalized T-rough prime and primary submodule of a module which is an extended notion of the lower and upper prime and primary submodule of a module. We hope that this extended research may provide a powerful tool in approximate reasoning. We believe that T-rough modules offered here will turn out to be more useful in the theory and applications of rough sets.

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