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On the Partial Ordering of s-Unitary Matrices

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Abstract

The concept of partial ordering on s-unitary matrices is introduced. Some results relating to partial ordering on s-unitary matrices are given.

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1 Introduction

Some characterizations of the star partial ordering and rank subtractivity for matrices was discussed by R.E.Hartwig and G.P.H.Styan in [4]. A relationship between star and minus partial ordering was settled by J.K.Baksalary in [3]. Jurgen Grob observed some remarks on partial ordering of hermitian matrices in [6]. In [5], Jorma K.Merikoski and Xiaogi Liu have developed star partial ordering on normal matrices. They have found several characterization s of $A \leq^* B$ in the case of the normal matrices. In this paper we introduce the concept of partial ordering on s-unitary matrices.

1.1 Preliminaries and Notations

Let $C_{n\times n}$ be the space of $n \times n$ complex matrices of order n. For $A \in C_{n\times n}$, let $A^T, \overline{A}, A^*, A^s, \overline{A}^s (A^\theta)$ denote transpose, conjugate, conjugate transpose, secondary transpose, conjugate secondary transpose of a matrix A respectively. Anna Lee[1] has initiated the study of secondary symmetric matrices. Also she has shown that for a complex matrix A, the usual transpose and secondary transpose A^T are related as $A^s = VA^TV$ Where 'V'is the permutation matrix whose elements on the secondary diagonal are 1 and other elements are zero. Also $\overline{A}^s$ denotes the conjugate secondary transpose of A. i.e $\overline{A}^s = (c_{ij})$ where $c_{ij} = \overline{a_{n-j+1,n-i+1}}$ [2] and $\overline{A}^s = VA^*V = A^\theta$. Also 'V'satisfies the following properties. $V^T = \overline{V} = V^* = V$ and $V^2 = I$.

A matrix $A \in C_{n\times n}$ is said to be s-hermitian if $A = A^\theta$ where $A^\theta = \overline{A}^s$ where $A^\theta = VA^*V$.

A matrix $A \in C_{n\times n}$ is said to be involutory if $A^2 = I$

A matrix $A \in C_{n\times n}$ is said to be normal if $AA^* = A^*A$

A matrix $A \in C_{n\times n}$ is said to be s-unitary if $AA^\theta = A^\theta A = I$.

That is $AVA^*V = VA^*VA = I$. That is $VA^*V = A^{-1}$.

In other words $VAA^* = A^*AV = V$.

For further results on s-unitary matrices one can refer [7].

2 Matrix Partial Ordering on Matrices

Definition 2.1. (Lowener Partial Ordering)
For $A, B \in C_{n\times n}$, A $\underset{L}{\geq}$ B if and only if $A - B \geq 0$.

Definition 2.2. (Star Partial Ordering)
For $A, B \in C_{n\times n}$, A $\underset{*}{\geq}$ B if and only if $B^*B = B^*A$ and $BB^* = AB^*$.

Definition 2.3. (Minus Partial Ordering)
For $A, B \in C_{n\times n}$, A $\underset{rs}{\geq}$ B if and only if $rank(A - B) = rankA - rankB$.

Remark 2.4. For $A, B \in C_{n\times n}$, A $\underset{rs}{\geq}$ B $\Leftrightarrow B = BA^{(1)}B = BA^{(1)}A = AA^{(1)}B$.

For proving some of the results in partial ordering we need the following theorem.

Theorem 2.5. *If A $\underset{L}{\geq}$ B for matrices A and B then $A - B$ is s-hermitian.*

Proof. A $\underset{L}{\geq}$ B $\Rightarrow$ VA $\underset{L}{\geq}$ VB
$VA - VB \geq 0 \Rightarrow V(A - B) \geq 0$. Hence $A - B$ is s-hermitian positive definite.
$(A - B)^* = V(A - B)V \Rightarrow (A - B)$ is s-hermitian. □

3 Partial Ordering of s-Unitary Matrices

In this section we have proved some results relating to partial ordering of s-unitary matrices. Also some of the theorems relating to lowener partial ordering and star partial ordering are given.

Theorem 3.1. *Let A and B are s-unitary matrices and hermitian. Then A $\underset{L}{\geq}$ B if and only if $A^{-1} \underset{L}{\geq} B^{-1}$.*

Proof. Assuming that A $\underset{L}{\geq}$ B
$\Rightarrow$ VA $\underset{L}{\geq}$ VB $\Rightarrow A^* \underset{L}{\geq} B^*$ (since A and B are hermitian)
Post multiplying by V, we get,
$VA^*V \underset{L}{\geq} VB^*V \Rightarrow A^{-1} \underset{L}{\geq} B^{-1}$.
Conversely, if $A^{-1} \underset{L}{\geq} B^{-1}$
$\Rightarrow VA^*V \underset{L}{\geq} VB^*V$ (since A and B are s-unitary matrices)
$\Rightarrow VA^* \underset{L}{\geq} VB^*$ (since A and B are hermitian)
$\Rightarrow VA \underset{L}{\geq} VB \Rightarrow$ A $\underset{L}{\geq}$ B □

Theorem 3.2. *Let A and B are s-unitary matrices and let A $\underset{L}{\geq}$ B. If A is involutary then B is also involutary and vice versa.*

Proof. A $\underset{L}{\geq}$ B $\Rightarrow$ VA $\underset{L}{\geq}$ VB $\Rightarrow VA - VB \geq 0 \Rightarrow V(A-B) \geq 0$
Hence $A-B$ is s-hermitian positive definite. Therefore $(A-B)$ is s-hermitian.

$$\begin{aligned} \Rightarrow (A-B)^* &= V(A-B)V \\ A^* - B^* &= V(A-B)V \\ VA^{-1}V - VB^{-1}V &= V(A-B)V = VAV - VBV \end{aligned}$$

Therefore

$$A^{-1} - B^{-1} = A - B \tag{3.1}$$

Assume that A is involutary. Premultiplying (3.1) by A,
$I - AB^{-1} = I - AB \Rightarrow AB^{-1} = AB$
Premultiplying by $A^{-1} \Rightarrow B^{-1} = B$
Premultiplying by $B \Rightarrow I = B^2$. Therefore B is involutary. If we assume B is involutary then we can show that A is involutary. □

Theorem 3.3. *Let A $\underset{L}{\geq}$ B. If A is s-unitary matrix and s- hermitian then B is s-hermitian.*

Proof. Since A $\underset{L}{\geq}$ B we have $(A-B)$ is s-hermitian. (By Theorem.2.5)

$$\begin{aligned} (A-B)^* &= V(A-B)V \\ [(A-B)^*]^* &= [V(A-B)V]^* = V^*(A-B)^*V^* \\ (A-B) &= V(A^* - B^*)V = VA^*V - VB^*V \\ \Rightarrow (A-B) &= A^{-1} - VB^*V \end{aligned}$$

Since A is s-hermitian $A = A^{-1}$. Therefore $B = VB^*V \Rightarrow$ B is s-hermitian. □

Theorem 3.4. *Let A and B are s-unitary matrices. Then* $A \underset{*}{\geq} B$ *if and only if* $A^{-1} \underset{*}{\geq} B^{-1}$.

Proof. Assuming that A $\underset{*}{\geq}$ B then we have (i). $B^*B = B^*A$ (ii). $BB^* = AB^*$
From (i), $VB^*BV = VB^*AV \Rightarrow VB^*VVBV = VB^*VVAV \Rightarrow (B^{-1})(VBV) = (B^{-1})VAV$
Taking (*) on bothsides, $(VBV)^*(B^{-1})^* = (VAV)^*(B^{-1})^*$
$\Rightarrow (V^*B^*V^*)(B^{-1})^* = (V^*A^*V^*)(B^{-1})^* \Rightarrow (VB^*V)(B^{-1})^* = (VA^*V)(B^{-1})^*$

$$\Rightarrow (B^{-1})(B^{-1})^* = (A^{-1})(B^{-1})^* \tag{3.2}$$

From (ii), $VBB^*V = VAB^*V \Rightarrow VBVVB^*V = VAVVB^*V$
$(VBV)(VB^*V) = (VAV)(VB^*V)$
Taking (*) on bothsides, $(VB^*V)^*(VBV)^* = (VB^*V)^*(VAV)^*$
$\Rightarrow (B^{-1})^*(V^*B^*V^*) = (B^{-1})^*(V^*A^*V^*) \Rightarrow (B^{-1})^*(VB^*V) = (B^{-1})^*(VA^*V)$

$$\Rightarrow (B^{-1})^*(B^{-1}) = (B^{-1})^*(A^{-1}) \tag{3.3}$$

From (3.2) and (3.3), we have, A $\underset{*}{\geq}$ B $\Rightarrow A^{-1} \underset{*}{\geq} B^{-1}$.
Similarly we can prove $A^{-1} \underset{*}{\geq} B^{-1} \Rightarrow$ A $\underset{*}{\geq}$ B. □

Theorem 3.5. *Let* $AV \underset{*}{\geq} VA$. *Then A is unitary if and only if A is s-unitary.*

Proof. AV $\underset{*}{\geq}$ VA $\Rightarrow (VA)^*VA = (VA)^*AV \Rightarrow A^*VVA = A^*VAV$

$$A^*A = A^*VAV \tag{3.4}$$

If A is unitary then from (3.4), $I = A^*VAV$
Premultiplying and Post multiplying (3.4) by V, $VA^*AV = VA^*VAVV$
$\Rightarrow I = VA^*VA$
Post multiplying (3.4) by $A^{-1} \Rightarrow A^{-1} = VA^*V$. Therefore A is s-unitary.
Conversely, if A is s-unitary then from (3.4), $A^*A = (VA^{-1}V)VAV = I$. A is unitary. □

Theorem 3.6. *Let* $A \underset{L}{\geq} B$ *and* $A \underset{*}{\geq} B$. *If A and B are s-unitary matrices then* $A = B$

Proof. A $\underset{*}{\geq}$ B $\Rightarrow$ (i). $B^*B = B^*A$ (ii). $BB^* = AB^*$
Therefore $B^*(A - B) = 0$. Taking (*) on bothsides, $[B^*(A - B)]^* = 0$

$$(A - B)^*B = 0 \tag{3.5}$$

Since A $\underset{L}{\geq}$ B we have $(A - B)^* = V(A - B)V$ (by Theorem 3.1)
Hence from (3.5), we have, $V(A - B)VB = 0$
$VAVB - VBVB = 0 \Rightarrow VAVB = VBVB$
Taking (*) on bothsides,

$$B^*V^*A^*V^* = B^*V^*B^*V^* \Rightarrow B^*VA^*V = B^*VB^*V$$
$$B^*A^{-1} = B^*B^{-1} \Rightarrow A^{-1} = B^{-1}$$

Therefore $A = B$ □

Theorem 3.7. *Let A and B are s-unitary matrices. Then $A \underset{rs}{\geq} B$ if and only if $A^{-1} \underset{rs}{\geq} B^{-1}$.*

Proof. Assume that A $\underset{rs}{\geq}$ B
$\Rightarrow B = BA^{(1)}A = AA^{(1)}B = BA^{(1)}B$ (by Remark 2.4)
$VBV = VBA^{(1)}AV = VAA^{(1)}BV = VBA^{(1)}BV$. Taking (*) on both sides,

$$
\begin{aligned}
(VBV)^* &= (VBA^{(1)}AV)^* = (VAA^{(1)}BV)^* = (VBA^{(1)}BV)^* \\
V^*B^*V^* &= V^*A^*(A^{(1)})^*B^*V^* = V^*B^*(A^{(1)})^*A^*V^* = V^*B^*(A^{(1)})^*B^*V^* \\
VB^*V &= VA^*(A^{(1)})^*B^*V = VB^*(A^{(1)})^*A^*V = VB^*(A^{(1)})^*B^*V \\
VB^*V &= VA^*(A^*)^{(1)}B^*V = VB^*(A^*)^{(1)}A^*V = VB^*(A^*)^{(1)}B^*V \\
VB^*V &= VA^*V(VA^*V)^{(1)}VB^*V = VB^*V(VA^*V)^{(1)}VA^*V \\
&= VB^*(VA^*V)^{(1)}VB^*V \\
B^{-1} &= A^{-1}(A^{-1})^{(1)}B^{-1} = B^{-1}(A^{-1})^{(1)}A^{-1} = B^{-1}(A^{-1})^{(1)}B^{-1}
\end{aligned}
$$

$\Rightarrow A^{-1} \underset{rs}{\geq} B^{-1}$. Therefore A $\underset{rs}{\geq}$ B $\Rightarrow A^{-1} \underset{rs}{\geq} B^{-1}$.
Similarly we can prove $A^{-1} \underset{rs}{\geq} B^{-1} \Rightarrow$ A $\underset{rs}{\geq}$ B. □

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