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Weak Annihilator Condition and Zip Rings

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Abstract

In [15], L. Ouyang introduced the notion of weak zip rings and proved that, a ring R is weak zip if and only if the skew polynomial ring $R[x;\alpha,\delta]$ is weak zip, when R is (α,δ)-compatible and reversible. We prove that, when R is (α,δ)-compatible and quasi-IFP in the sense of Kim et al. [9], then $ni\ell(R)[x;\alpha,\delta] = ni\ell(R[x;\alpha,\delta])$, and R is weak zip if and only if the skew polynomial ring $R[x;\alpha,\delta]$ is weak zip.

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1 Introduction

Throughout, R denotes an associative ring with unity, $\alpha : R \to R$ is an endomorphism and $\delta : R \to R$ an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b+\alpha(a)\delta(b)$, for all $a, b \in R$. We adopt the notations $M_n(R)$, $T_n(R)$ and $R[x]$ to represent the ring of all $n \times n$ full matrices, the ring of all $n \times n$ upper triangular matrices and the ring of all polynomials over a ring R in indeterminate x, respectively. We denote by $R[x;\alpha,\delta]$ the Ore extension whose elements are the polynomials over R, the addition is defined as usual, and the multiplication subject to the relation $xa = \alpha(a)x + \delta(a)$, for each $a \in R$. By a $ni\ell(R)$, we mean the set of all nilpotent elements in R. Recall that a ring R is called *reduced* if $a^2 = 0$ implies that $a = 0$, for all $a \in R$; R is *symmetric* if $abc = 0$ implies $acb = 0$, for all $a, b, c \in R$; R is *reversible* if $ab = 0$ implies $ba = 0$, for all $a, b \in R$; R has IFP (or is *semicommutative*) if $ab = 0$ implies $aRb = 0$, for all $a, b \in R$. A ring R is called 2-*primal* if $P(R) = ni\ell(R)$, where $P(R)$ is the prime radical of R.

According to Krempa [12], an endomorphism α of a ring R is called *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is α-*rigid* if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced by Hong et al. [7]. Properties of α-rigid rings have been studied in Krempa [12], Hong et al. [7] and others. By Hashemi and Moussavi [6], a ring R is said to be *α-compatible* if for each $a, b \in R, ab = 0 \Leftrightarrow a\alpha(b) = 0$. Moreover, R is called *δ-compatible* if for each $a, b \in R, ab = 0 \Rightarrow a\delta(b) = 0$. If R is both α-compatible and δ-compatible, then R is said to be *(α, δ)-compatible*. By [6], a ring R is α-rigid if and only if R is (α, δ)-compatible and reduced.

H. K. Kim et al. [9] introduced the notion of quasi-*IFP*, and defined a ring R to have *quasi-IFP* provided that $\sum_{i=0}^{n} Ra_iR$ is nilpotent whenever $\sum_{i=0}^{n} a_i x^i \in R[x]$ is nilpotent. They proved that, a ring R has quasi-IFP if and only if $N_0(R) = ni\ell(R)$, where $N_0(R)$ is the Wedderburn radical of R. They also showed that every ring with quasi-*IFP* is 2-primal, and rings with *IFP* (semicommutative rings) have quasi-*IFP*. In general, each of these implications is irreversible.

Faith [4] called a ring R *right zip* provided that if the right annihilator $r_R(X)$ of a subset X of R is zero, then there exists a finite subset $Y \subseteq X$ such that $r_R(Y) = 0$. R is *zip* if it is right and left zip. In [15], Ouyang introduced the notion of weak zip rings and investigated their properties. Put $Nr_R(X) = \{a \in R \mid xa \in ni\ell(R) \text{ for all } x \in X\}$ and $Nl_R(X) = \{a \in R \mid ax \in ni\ell(R) \text{ for all } x \in X\}$. It is not hard to see that weak zip property is left-right symmetric. Ouyang [15] proved that for an endomorphism α and α-derivation δ of a ring R, R is right (left) weak zip if and only if the skew polynomial ring $R[x; \alpha, \delta]$ is right (left) weak zip, when R is (α, δ)-compatible and reversible. We extend the results in [15] and show that when R is (α, δ)-compatible and quasi-IFP, then $ni\ell(R)[x; \alpha, \delta] = ni\ell(R[x; \alpha, \delta])$, and that R is a weak zip ring if and only if the skew polynomial ring $R[x; \alpha, \delta]$ is weak zip.

2 Main Results

For any $0 \leq i \leq j$, $f_i^j \in End(R, +)$ will denote the map which is the sum of all possible "words" in α, δ built with i letters α and $j - i$ letters δ (e.g., $f_n^n = \alpha^n$, $f_0^n = \delta^n, \dots,$ $f_{n-1}^n = \alpha^{n-1}\delta + \alpha^{n-2}\delta\alpha + \cdots + \delta\alpha^{n-1}$). Properties of the right (left) annihilator of a subset in a ring R are studied by many authors. As a generalization of the right (left) annihilator, in [15] Ouyang introduced the notion of a weak annihilator of a subset in a ring, and investigated the weak annihilator properties over the Ore extension ring $R[x; \alpha, \delta]$. Motivated by the results in [15], in this section, we continue the study of weak zip property of Ore extension rings.

Lemma 2.1. *Let R be an (α, δ)-compatible ring such that $ni\ell(R)$ is an ideal*

of R. Then, for any $a, b, c \in ni\ell(R)$, we have the following:
(1) If $abc = 0$, then $ab\delta^s(c) = 0$ and $a\delta^t(b)c = 0$, for all positive integers s, t.
(2) If $abc = 0$, then $a\delta^s(b)\delta^t(c) = 0$, for all positive integers s, t.
(3) If $ab = 0$, then $\alpha^n(a)\delta^m(b) = 0, \delta^m(a)\alpha^n(b) = 0$, for all positive integers m, n.

Proof. (1) Since R is δ-compatible, $\delta(ni\ell(R)) \subseteq ni\ell(R)$, so we have $ab\delta^s(c) = 0$. From $abc = 0$, since $ni\ell(R)$ is an ideal, we get $a\delta(bc) = a\delta(b)c + ab\delta(c) = 0$. Thus $a\delta(b)c = 0$, so $a\delta(b)\delta(c) = 0$. So $a\delta(\delta(b)c) = a\delta^2(b)c + a\delta(b)\delta(c) = 0$ and hence $a\delta^2(b)c = 0$. Continuing in this process we get $a\delta^t(b)c = 0$.
(2) Since $\delta(ni\ell(R)) \subseteq ni\ell(R)$, it is an immediate consequence of (1).
(3) From (1), we obtain $\alpha^n(a)\delta^m(b) = 0$. From $ab = 0$, we get $\delta(a)b + a\delta(b) = 0$. Thus $\delta(a)b = 0$, since $ni\ell(R)$ is an ideal of R. So, $\delta^2(a)b + \alpha(\delta(a))\delta(b) = 0$. Hence $\alpha(\delta(a))\delta(b) = 0$ and thus $\delta^2(a)b = 0$. Continuing in this process, we get $\delta^m(a)b = 0$ and hence $\delta^m(a)\alpha^n(b) = 0$, as needed.

According to [9] a ring R is called *quasi-IFP*, provided that $\sum_{i=0}^n Ra_iR$ is nilpotent whenever $\sum_{i=0}^n a_ix^i \in R[x]$ is nilpotent. Also, by [9, Lemma 1.3], If R is a quasi-IFP ring, then $ni\ell(R)$ forms an ideal of R.

Theorem 2.2. *Let R be an (α, δ)-compatible quasi-IFP ring. Then we have,*

$$ni\ell(R)[x; \alpha, \delta] = ni\ell(R[x; \alpha, \delta]).$$

Proof. For the forward direction, suppose that $a_i \in ni\ell(R)$ for each $0 \le i \le n$. Then there exist k_i such that

$$(a_iR)^{k_i} = 0, \quad 0 \le i \le n. \tag{1}$$

Let $k = k_0 + k_1 + \cdots + k_n + 1$. Then we have $(a_iR)^k = 0$, for each $0 \le i \le n$. Taking $f(x) = a_0 + a_1x + \cdots + a_nx^n$, we claim that $[f(x)]^k = 0$. We have $\left(\sum_{i=0}^n a_ix^i\right)^2 = \sum_{i=0}^n a_if_0^i(a_0) + \left(\sum_{i=1}^n a_if_1^i(a_0) + \sum_{i=0}^n a_if_0^i(a_1)\right)x +$
$\left(\sum_{i=2}^n a_if_2^i(a_0) + \sum_{i=1}^n a_if_1^i(a_1) + \sum_{i=0}^n a_if_0^i(a_2)\right)x^2 + \cdots +$
$\left(\sum_{s+t=k}\left(\sum_{i=s}^n a_if_s^i(a_t)\right)\right)x^k + \cdots + a_n\alpha^n(a_n)x^{2n}$. We can show that the coefficients of $\left(\sum_{i=0}^n a_ix^i\right)^k$ can be written as sums of monomials of length k in a_i and $f_u^v(a_j)$, where $a_i, a_j \in \{a_0, a_1, ..., a_n\}$ and $v \ge u \ge 0$ are positive integers. Consider each monomial

$$a_{i_1}(f_{s_2})^{t_2}(a_{i_2})..(f_{s_k})^{t_k}(a_{i_k}) = 0,$$

where $a_{i_1}, a_{i_2}, \ldots, a_{i_k} \in \{a_0, a_1, ..., a_n\}$ and t_j, s_j $(t_j \ge s_j, 2 \le j \le k)$ are non-negative integers. We will show that $a_{i_1}(f_{s_2})^{t_2}(a_{i_2})..(f_{s_k})^{t_k}(a_{i_k})$=0. If the number of a_0 in $a_{i_1}(f_{s_2})^{t_2}(a_{i_2})..(f_{s_k})^{t_k}(a_{i_k})$ is greater than k_o, then we can write monomial $a_{i_1}(f_{s_2})^{t_2}(a_{i_2})..(f_{s_k})^{t_k}(a_{i_k})$ as

$$b_1(f_{s_{01}}^{t_{01}}(a_0))^{j_1}b_2(f_{s_{02}}^{t_{02}}(a_0))^{j_2}\cdots b_v(f_{s_{0v}}^{t_{0v}}(a_0))^{j_v}b_{v+1},$$

where $j_1 + j_2 + \cdots + j_v > k_0$, $1 \leq j_1, j_2, ..., j_v$ and b_q $(q = 1, 2, ..., v+1)$ is a product of some elements choosing from $\{a_{i_1}, f_{s_2}^{t_2}(a_{i_2}), ..., f_{s_k}^{t_k}(a_{i_k})\}$ or is equal to 1. Since by (1) we have $(Ra_iR)^{k_i} = 0$, so $(a_0)^{j_1}b_2(a_0)^{j_2} \cdots b_v(a_0)^{j_v}b_{v+1} = a_0...a_0b_2a_0...a_0b_3 \cdots b_va_0...a_0b_{v+1} = 0$, since $b_qa_0, a_0, a_0b_q \in a_0R, (q = 1, 2, ..., v+1)$ and thus we have $b_1(f_{s_{01}}^{t_{01}}(a_0))^{j_1}b_2(f_{s_{02}}^{t_{02}}(a_0))^{j_2} \cdots b_v(f_{s_{0v}}^{t_{0v}}(a_0))^{j_v}b_{v+1} = 0$. Thus $a_{i_1}f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_k}^{t_k}(a_{i_k}) = 0$. If the number of a_i in $a_{i_1}f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_k}^{t_k}(a_{i_k})$ is greater than k_i, then a similar discussion yields that $a_{i_1}f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_k}^{t_k}(a_{i_k})$ =0. Thus each monomial appears in $(\sum_{i=0}^n a_ix^i)^k$ equals to 0. Therefore $\sum_{i=0}^n a_ix^i \in R[x; \alpha, \delta]$ is a nilpotent element, as desired.

For the backward direction, suppose that $f(x) \in nil(R[x; \alpha, \delta])$. There exists some positive integer $k = 2^p$ such that $[f(x)]^k = (a_0+a_1x+\cdots+a_nx^n)^k = 0$. Then $0 = [f(x)]^k = \textit{lower terms} + a_n\alpha^n(a_n)\alpha^{2n}(a_n)\ldots\ \alpha^{(k-1)n}(a_n)x^{nk}$. Hence $a_n\alpha^n(a_n)\cdots\alpha^{(k-1)n}(a_n) = \left(a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(\frac{k}{2}-1)n}(a_n)\right)$. Therefore $\alpha^{\frac{kn}{2}}\left(a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(\frac{k}{2}-1)n}(a_n)\right) = 0$. Since R is (α, δ)-compatible ring, then $a_n\alpha^n(a_n)\ \alpha^{2n}(a_n)\cdots\alpha^{(\frac{k}{2}-1)n}(a_n) \in nil(R)$. On the other hand, we can see $\left(a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(\frac{k}{4}-1)n}(a_n)\right)\alpha^{\frac{kn}{4}}\left(a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(\frac{k}{4}-1)n}(a_n)\right)$ is nilpotent, since R is (α, δ)-compatible, then $a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(\frac{k}{4}-1)n}(a_n) \in nil(R)$. Applying the preceding method repeatedly, we have $a_n \in nil(R)$. Then $f_i^j(a_n) \in nil(R)$ for all $0 \leq i \leq j$. Let $Q = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$. Then we have $0 = (Q + a_nx^n)^k = (Q + a_nx^n)(Q + a_nx^n)\ldots(Q + a_nx^n) = (Q^2 + Q \cdot a_nx^n + a_nx^n \cdot Q + a_nx^n \cdot a_nx^n)(Q + a_nx^n)\ldots(Q + a_nx^n) = \ldots = Q^k + \Delta$, where $\Delta \in R[x; \alpha, \delta]$. Note that the coefficients of Δ can be written as sums of monomials in a_i and $f_u^v(a_j)$ where $a_i, a_j \in a_0, a_1, \ldots, a_n$ and $v \geq u \geq 0$ are positive integers, and each monomial has an or $f_s^t(a_n)$. Since $nil(R)$ is an ideal, we obtain that each monomial is in $nil(R)$, and so $\Delta \in nil(R)[x; \alpha, \delta]$. Thus we obtain $(a_0+a_1x+\cdots+a_{n-1}x^{n-1})^k = \textit{lower terms} + a_{n-1}\alpha^{n-1}(a_{n-1})\ldots\alpha^{(n-1)(k-1)}(a_{n-1})x^{(n-1)k} \in nil(R)[x; \alpha, \delta]$, since $nil(R)$ is an ideal of R. Hence $a_{n-1}\alpha^{n-1}(a_{n-1})\ldots\alpha^{(k-1)(n-1)}(a_{n-1}) \in nil(R)$ and so $a_{n-1} \in nil(R)$, since R is (α, δ)-compatible. Using induction on n, we obtain $a_i \in nil(R)$ for all $0 \leq i \leq n$, and the result follows.

Corollary 2.3. [15, Lemma 3.6.] *Let R be an (α, δ)-compatible and reversible. Then,*

$$nil(R)[x; \alpha, \delta] = nil(R[x; \alpha, \delta]).$$

Theorem 2.4. *Let R be a ring such that $nil(R)[x; \alpha, \delta] = nil(R[x; \alpha, \delta])$. If $nil(R)$ is an ideal of R, then $ab \in nil(R)$ if and only if $af_i^j(b) \in nil(R)$.*

Proof. Since $ab \in nil(R)$, so $ba \in nil(R)$. Assume that $f = b$ and $g = ax \in R[x; \alpha, \delta]$. Then $fg \in nil(R[x; \alpha, \delta])$, so $gf = a\delta(b) + a\alpha(b)x \in nil(R[x; \alpha, \delta])$. Thus $a\delta(b), a\alpha(b) \in nil(R)$. Now suppose that $f = \alpha(b), g = ax \in R[x; \alpha, \delta]$, then $fg \in ni\ell(R[x;\alpha,\delta])$. So $gf = a\delta\alpha(b) + a\alpha^2(b)x \in ni\ell(R[x;\alpha,\delta])$, and hence $a\delta\alpha(b), a\alpha^2(b) \in ni\ell(R)$. Since $a\delta(b) \in ni\ell(R)$, for $h = \delta(b)$ and $k = ax \in R[x;\alpha,\delta]$, we have $hk \in ni\ell(R[x;\alpha,\delta])$, so $kh = a\delta^2(b) + a\alpha\delta(b)x \in ni\ell(R[x;\alpha,\delta])$, thus $a\alpha\delta(b), a\delta^2(b) \in ni\ell(R)$. Continuing in this process we obtain $a\alpha^{k_1}\delta^{t_1}\alpha^{k_2}\delta^{t_2}\ldots\alpha^{k_p}\delta^{t_j}(b) \in ni\ell(R)$, where $k_i, t_j \geq 0$. Thus $af_i^j(b) \in ni\ell(R)$.

Corollary 2.5. *Let R be an (α,δ)-compatible quasi-IFP ring. Then $ab \in ni\ell(R)$ if and only if $af_i^j(b) \in ni\ell(R)$.*

Lemma 2.6. *Let R be an (α,δ)-compatible quasi-IFP ring. If $f(x) = \sum_{i=0}^m a_ix^i, g(x) = \sum_{j=0}^n b_jx^j \in R[x;\alpha,\delta]$ such that $f(x)g(x) \in ni\ell(R[x;\alpha,\delta])$, then $a_ib_j \in ni\ell(R)$, for each $0 \leq i \leq m$ and $0 \leq j \leq n$.*

Proof. We have $f(x)g(x) = (\sum_{i=0}^m a_ix^i)(\sum_{j=0}^n b_jx^j) = (\sum_{i=0}^m a_ix^i)\, b_0 + (\sum_{i=0}^m a_ix^i)\, b_1x + \cdots + (\sum_{i=0}^m a_ix^i)\, b_nx^n = \sum_{i=0}^m a_if_0^i(b_0) + (\sum_{i=1}^m a_if_1^i(b_0) + \sum_{i=0}^m a_if_0^i(b_1))\, x + \cdots + (\sum_{s+t=k}(\sum_{i=s}^m a_if_s^i(b_t)))\, x^k + \cdots + a_m\alpha^m(b_n)x^{m+n}$. From Theorem 2.2, we have the following equations:

$$\Delta_{m+n} = a_m\alpha^m(b_n) \in ni\ell(R) \tag{2}$$

$$\Delta_{m+n-1} = a_m\alpha^m(b_{n-1}) + a_{m-1}\alpha^{m-1}(b_n) + a_mf_{m-1}^m(b_n) \in ni\ell(R) \tag{3}$$

$$\Delta_{m+n-2} = a_m\alpha^m(b_{n-2}) + \sum_{i=m-1}^m a_if_{m-1}^i(b_{n-1}) + \sum_{i=m-2}^m a_if_{m-2}^i(b_n) \in ni\ell(R) \tag{4}$$

$$\sum_{s+t=k}(\sum_{i=s}^m a_if_s^i(b_t)) \in ni\ell(R). \tag{5}$$

From equation (2) and Corollary 2.5, we obtain $a_mb_n \in ni\ell(R)$ $(*)$, so $b_na_m \in ni\ell(R)$. If we multiply equation (3) on the left side by b_n, then $b_na_{m-1}\alpha^{m-1}(b_n) \in ni\ell(R)$, since $ni\ell(R)$ is an ideal. Thus by Corollary 2.5, we obtain $b_na_{m-1}b_n \in ni\ell(R)$, and so $b_na_{m-1} \in ni\ell(R), a_{m-1}b_n \in ni\ell(R)$ $(**)$. From $(*), (**)$ and the fact that $\Delta_{m+n-1} \in ni\ell(R)$ and $ni\ell(R)$ is an ideal, using Corollary 2.5, we have $a_m\alpha^m(b_{n-1}) \in ni\ell(R)$, thus $a_mb_{n-1} \in ni\ell(R)$ $(***)$. Now, multiply equation (4) from right on the a_m. Since $ni\ell(R)$ is an ideal, we deduce that $a_m\alpha^m(b_{n-2}) \in ni\ell(R)$. We also have $\sum_{i=m-1}^m a_if_{m-1}^i(b_{n-1}) + \sum_{i=m-2}^m a_if_{m-2}^i(b_n) = a_{m-1}f_{m-1}^{m-1}(b_{n-1}) + a_mf_{m-1}^m(b_{n-1}) + a_{m-2}f_{m-2}^{m-2}(b_n) + a_{m-1}f_{m-2}^{m-1}(b_n) + a_mf_{m-2}^m(b_n) \in ni\ell(R)$, since $ni\ell(R)$ is an ideal. By $(*), (**), (***)$, Corollary 2.5 and the fact that $ni\ell(R)$ is an ideal, we get $a_{m-1}f_{m-1}^{m-1}(b_{n-1}) + a_{m-2}f_{m-2}^{m-2}(b_n) \in ni\ell(R)$. $(****)$ Multiply equation (4) from right on the a_{m-1} and using $(**)$, we get $a_{m-1}f_{m-1}^{m-1}(b_{n-1})a_{m-1} \in ni\ell(R)$, by

Corollary 2.5, $a_{m-1}b_{n-1} \in ni\ell(R)$. Using $(****)$ and that $ni\ell(R)$ is an ideal, we deduce that $a_{m-2}b_n \in ni\ell(R)$. Applying the preceding method repeatedly, we deduce that $a_ib_j \in ni\ell(R)$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$.

Theorem 2.7. *Let R be an (α, δ)-compatible quasi-IFP ring. Then R is weak zip if and only if $R[x; \alpha, \delta]$ is weak zip.*

Proof. Suppose that R is a weak zip ring and $S = R[x; \alpha, \delta]$. Let $X \subseteq S$ such that $Nr_S(X) \subseteq ni\ell(S)$. For $f(x) = \sum_{i=0}^{n} a_ix^i \in S, C_f$ denotes the set of coefficients of $f(x)$, and for a subset V of S, C_V denotes the set $\bigcup_{f \in V} C_f$. Then $C_V \subseteq R$. Now, we show that $Nr_R(C_X) \subseteq ni\ell(R)$. If $r \in Nr_R(C_X)$, then $ar \in ni\ell(R)$ for any $a \in C_X$. So for any $f(x) = \sum_{i=0}^{n} a_ix^i \in X$, we obtain $a_ir \in ni\ell(R)$ and so $a_if_s^t(r) \in ni\ell(R)$, by Corollary 2.5. Hence $f(x)r \in ni\ell(S)$, by Theorem 2.2. So $r \in Nr_S(X) \subseteq ni\ell(S)$ and thus $r \in ni\ell(R)$ and $Nr_R(C_X) \subseteq ni\ell(R)$. Since R is weak zip, there exists a finite subset $Y_0 \subseteq C_X$ such that $Nr_R(Y_0) \subseteq ni\ell(R)$. For each $a \in Y_0$, there exists $g_a(x) \in X$ such that some of the coefficients of $g_a(x)$ are a. Let X_0 be a minimal subset of X such that $g_a(x) \in X_0$ for each $a \in Y_0$. Then X_0 is a finite subset of X. Let Y_1 be the set of all coefficients of elements of X_0, then $Y_0 \in Y_1$ and so $Nr_R(Y_1) \subseteq Nr_R(Y_0) \subseteq ni\ell(R)$. If $f(x) = a_0 + a_1x + \cdots + a_kx^k \in Nr_S(X_0)$, then $g(x)f(x) \in ni\ell(S)$ for any $g(x) = b_0 + b_1x + \cdots + b_tx^t \in X_0$. By Lemma 2.6, we obtain $b_ia_j \in ni\ell(R)$ for each i, j. Thus $a_j \in Nr_R(Y_1) \subseteq ni\ell(R)$ for $0 \leq j \leq k$ and so $f(x) \in ni\ell(S)$, by Theorem 2.2. Hence $Nr_S(X) \subseteq ni\ell(S)$. Therefore, $R[x; \alpha, \delta]$ is a weak zip ring. Conversely, suppose that $S = R[x; \alpha, \delta]$ is weak zip. Let Y be a subset of R such that $Nr_R(Y) \subseteq ni\ell(R)$. If $f(x) = a_0 + a_1x + \cdots + a_nx^n \in Nr_S(Y)$, then $a_i \in Nr_R(Y) \subseteq ni\ell(R)$ for all $0 \leq i \leq n$, and so $f(x) \in ni\ell(S)$, by Theorem 2.2 and hence $Nr_S(Y) \subseteq ni\ell(S)$. Since $S = R[x; \alpha, \delta]$ is weak zip, there exists a finite subset $Y_0 \subseteq Y$ such that $Nr_S(Y_0) \subseteq ni\ell(S)$. Hence, $Nr_R(Y_0) = Nr_S(Y_0) \cap R \subseteq ni\ell(R)$. Therefore, R is weak zip, and the result follows.

Corollary 2.8. [15, Theorem 3.11.] *Let R be an (α, δ)-compatible reversible ring. Then R is weak zip if and only if $R[x; \alpha, \delta]$ is weak zip.*

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