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FI-Injective Resolutions and Dimensions

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Abstract

In this paper, we show the existence of FI-injective preenvelopes over coherent rings, characterize FI-injective resolutions and define FI-injective dimension for modules and rings. It measures how far away a left module from being FI-injective module.

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1 Introduction

We first recall some known notions and facts which we need in the sequel.

Let R be a ring. A left R-module M is called FP-injective [9] if $Ext^1_R(N, M) = 0$ for all finitely presented left R-modules N.A left R-module M is called FI-injective [8] if $Ext^1_R(N, M) = 0$ for any FP-injective left R-module N.Let $\mathcal{C}$ be a class of right R-modules. Following [4], A homomorphism $\phi : M \to F$ with $F \in \mathcal{C}$ is called a $\mathcal{C}$-preenvelope of M, if for any homomorphism $f : M \to F'$ with $F' \in \mathcal{C}$, there is a homomorphism $g : F \to F'$ such that $g\phi = f$. Moreover, if the only such g are automorphisms of F when $F' = F$ and $f = \phi$, the $\mathcal{C}$-preenvelope ϕ is called a $\mathcal{C}$-envelope of M. Dually, we have the definitions of a $\mathcal{C}$-precover and a $\mathcal{C}$-cover.

A pair $(\mathcal{F}, \mathcal{C})$ of classes of R-modules is called a cotorsion theory if $\mathcal{F}^\perp = \mathcal{C}$ and ${}^\perp\mathcal{C} = \mathcal{F}$, where $\mathcal{F}^\perp = \{C : Ext^1_R(F, C) = 0 \text{ for all } F \in \mathcal{F}\}$, and ${}^\perp\mathcal{C} = \{F : Ext^1_R(F, C) = 0 \text{ for all } C \in \mathcal{C}\}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called perfect if every R-module has a $\mathcal{C}$-envelope and an $\mathcal{F}$-cover.

In what follows, let $\mathcal{FI}$ ($\mathcal{FP}$) be the class of all FI-injective (FP-injective) left R-modules.

In section 2, we show that if R is a left coherent and left self-FP-injective ring, then $(\mathcal{FP}, \mathcal{FI})$ is a perfect cotorsion theory. In particular, every left R-module has a special $\mathcal{FI}$-preenvelope, and every left R-module has a special $\mathcal{FP}$-precover.

In section 3, the definition and some general results are given. For a left R-module M, we define the FI-injective dimension $fid(M)$ of M to the smallest integer $n \geq 0$ such that $Ext_R^{n+1}(N, M) = 0$ for any FP-injective left R-module N. The left FI-injective dimension $lfiD(R)$ of a ring R is define as $sup\{fid(M) : M$ is any left R-module$\}$. It is shown that for a left coherent ring R, $lfiD(R) = sup\{fid(M) : M$ is any left R-module$\} = sup\{pd(F) : F$ is an FP-injective left R-module$\} = sup\{fid(F) : F$ is an FP-injective left R-module$\}$.

Throughout this article, R is an associative ring with identity and all modules are unitary. For an R-module M, the character module $Hom_Z(M, Q/Z)$ is denoted by M^+, $fd(M)$, $id(M)$ and $FP - id(M)$ denote the flat, injective and FP-injective dimension of M, respectively. General background materials can be found in [2] and [4].

2 FI-injective Envelopes

Lemma 2.1 *([5]) Let R be a left coherent ring and M a left R-module. Then $fd(M^+) = FP - id(M)$.*

Lemma 2.2 *([1]) Let $\mathcal{F}$ be a class of modules closed under direst sums, extensions, continuous well-ordered unions, and contain all projective modules. If $\mathcal{F}^{\perp} = S^{\perp}$ for a set $S \subseteq \mathcal{F}$, then $(\mathcal{F}, \mathcal{F}^{\perp})$ is a cotorsion theory.*

Recall that a ring R is called left self-FP-injective [9] if ${}_RR$ is an FP-injective module. In what follows, let $\mathcal{FI}$ ($\mathcal{FP}$) be the class of all FI-injective (FP-injective) left R-modules. Now we have the following theorem

Theorem 2.3 *If R is a left coherent and left self-FP-injective ring, then $(\mathcal{FP}, \mathcal{FI})$ is a perfect cotorsion theory. In particular, every left R-module has a special $\mathcal{FI}$-preenvelope, and every left R-module has a special $\mathcal{FP}$-precover.*

Proof: Let Card$(R) = \aleph_\beta$ and $F \in \mathcal{FP}$. By [4, Lemma 5.3.12], for each $x \in F$, there is a pure submodule S of F with $x \in S$ such that Card$(S) \leq \aleph_\beta$ (simply let $N = Rx$ and $f = id_N$ in the Lemma). So we can write F as a union of a continuous chain $(F_\alpha)_{\alpha<\lambda}$ of pure submodules of F such that Card$(F_0) \leq \aleph_\beta$ and Card$(F_{\alpha+1}/F_\alpha) \leq \aleph_\beta$ whenever $\alpha + 1 < \lambda$. If N is a left R-module such that $Ext_R^1(F_0, N) = 0$ and $Ext_R^1(F_{\alpha+1}/F_\alpha, N) = 0$ whenever $\alpha + 1 < \lambda$, then $Ext_R^1(F, N) = 0$ by[4, Theorem 7.3.4]. Since F_α is a pure submodule of F for any $\alpha < \lambda$, $F^+ \to F_\alpha^+ \to 0$ is split. Then $F_\alpha^+ \in \mathcal{F}$ since $F^+ \in \mathcal{F}$ by Lemma 2.1, and so $F_\alpha \in \mathcal{FP}$ by Lemma 2.1 again. On the other hand, F_α is a pure submodule of $F_{\alpha+1}$ whenever $\alpha + 1 < \lambda$, so the exact sequence $0 \to F_\alpha \to F_{\alpha+1} \to F_{\alpha+1}/F_\alpha \to 0$ induces the split exact sequence $0 \to (F_{\alpha+1}/F_\alpha)^+ \to F_{\alpha+1}^+ \to F_\alpha^+ \to 0$. Thus $(F_{\alpha+1}/F_\alpha)^+ \in \mathcal{F}$ since $F_{\alpha+1}^+ \in \mathcal{F}$ by Lemma 2.1, and hence $F_{\alpha+1}/F_\alpha \in \mathcal{FP}$. Let X be a set of representatives of all modules $G \in \mathcal{FP}$ with Card$(G) \leq \aleph_\beta$. Then $\mathcal{FI} = X^\perp$.

We note that $\mathcal{FP}$ is closed under direct sums, extensions, direct limits since R is left coherent, and contains all projective modules since R is self-FP-injective ring. Therefore $(\mathcal{FP}, \mathcal{FI})$ is a cotorsion theory by Lemma 2.2.

Since $(\mathcal{FP}, \mathcal{FI})$ is cogenerated by the set X, $(\mathcal{FP}, \mathcal{FI})$ is a complete cotorsion theory by [3, Theorem 10]. Moreover, $(\mathcal{FP}, \mathcal{FI})$ is a perfect cotorsion theory by [4, Theorem 7.2.6] (for $\mathcal{FP}$ is closed under direct limits).

Remark 2.4 *(1) We note that Theorem 2.3 extends the work of [1, Theorem 2.8], where the same result is obtained under the hypothesis that R is left Noetherian.*

(2) Choose a field F, and set $F_i = F$ for $i = 1, 2, \cdots$, $S = \prod_{i=1}^{\infty} F_i$. Then S is a commutative Von Neumann regular ring. Let $R = S$, the ring of polynomials indeterminates over S, then R is a coherent ring with $wD(R) = 0$ (see [6]). Clearly, the ring R satisfies the condition of Theorem 2.3, but it is not Noetherian.

3 FI-injective Dimensions

Definition 3.1 *Let R be a ring. For a left R-module M, let $fid(M)$ denote the smallest integer $n \geq 0$ such that $Ext_R^{n+1}(N, M) = 0$ for any FP-injective left R-module N and call $fid(M)$ the FI-injective dimension of M. If no such n exists, set $fid(M) = \infty$.*

Put $lfiD(R) = sup\{fid(M) : M$ is any left R-module$\}$, and call $fid(R)$ the left FI-injective dimension of R. Similarly, we have $rfid(R)$. If $fid(M) = 0$, then M is FI-injective module.

Remark 3.2 *(1)We also use $\mathcal{FI}$-preenvelope to characterize FI-injective dimension, i.e, $fid(M) = inf\{n :$ there is an $\mathcal{FI}$-preenvelope resolution of the form $0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to 0$ of $M\}$.*

(2)It is clear that $fid(M) \leq id(M)$ for any left R-module M and $lfid(R) \leq lD(R)$ for any ring R. It is also easy to see that a ring R is semihereditary if and only if $fid(M) = id(M)$ for all left R-module M if and only if every FI-injective left R-module is injective.

Proposition 3.3 *Let R be a left coherent ring. Then the following are equivalent for any left R-module M and an integer $n \geq 0$:*

(1) $fid(M) \leq n$.

(2) $Ext^i_R(N, M) = 0$ for any FP-injective left R-module N and $i \geq n+1$.

(3) Every n-th cosyzygy of M is FI-injective.

(4) There exists an exact sequence $0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to 0$, where each E^i is FI-injective.

Proof: (1) ⇔ (2) by definition.

(2) ⇔ (3) We simply note that if L^n is n-th cosyzygy of M, then $Ext^{j+n}_R(N, M) \cong Ext^j_R(N, L^n)$, so the result follows.

(1) ⇔ (4) is straightforward.

Corollary 3.4 *If R is noetherian ring and $id(M) < \infty$, then $fid(M) = id(M)$.*

Proof: $fid(M) \leq id(M)$ follows from Remark 3.2. Since injectives are FI-injectives. Now suppose $fid(M) = n$. Then $Ext^i_R(N, M) = 0$ for all FP-injectives N and $i \geq n+1$. But $id(M) < \infty$, so $id(M) \leq n$ by [7, Lemma 2.2].

Proposition 3.5 *Let $0 \to M \to E^0 \to E^1 \to \cdots$ be a FI-injective resolution of an R-module M. Then the sequence $0 \to T \bigotimes_R M \to T \bigotimes_R E^0 \to T \bigotimes_R E^1 \to \cdots$ is exact for all right strongly FI-flat R-module T.*

Proof: Let $K^i = ker(E^i \to E^{i+1}), i \geq 1$ and $K^0 = M$. We consider the short exact sequence $0 \to K^i \to E^i \to K^{i+1} \to 0$. Then $0 \to T \bigotimes_R K^i \to T \bigotimes_R E^i \to T \bigotimes_R K^{i+1} \to 0$ is exact if and only if $0 \to (T \bigotimes_R K^{i+1})^+ \to (T \bigotimes_R E^i)^+ \to (T \bigotimes_R K^i)^+ \to 0$ is exact. But the latter is equivalent to $0 \to Hom_R(K^{i+1}, T^+) \to Hom_R(E^i, T^+) \to Hom_R(K^i, T^+) \to 0$ being exact. So the result follows since T^+ is strongly FI-injective by [8, Remark 2.2(2)].

Proposition 3.6 *Let R be a left coherent ring, $0 \to A \to B \to C \to 0$ an exact sequence of left R-modules. If two of $fid(A), fid(B)$ and $fid(C)$ are finite, so is the third. Moreover,*

(1) $fid(B) \leq sup\{fid(A), fid(C)\}$.

(2) $fid(A) \leq sup\{fid(B), fid(C)+1\}$.

(3) $fid(C) \leq sup\{fid(B), fid(A)-1\}$.

Proof: By Definition and Proposition 3.3.

Corollary 3.7 *Let R be a left coherent ring. Then the following hold:*

(1) If $0 \to A \to B \to C \to 0$ is an exact sequence of left R-modules, where $1 < fid(A) < \infty$ and B is FI-injective, then $fid(C) = fid(A) - 1$.

(2) $rfiD(R) = n$ if and only if $sup\{fid(I) : I$ is any left ideal of $R\} = n+1$ for any integer $n \geq 0$.

Proof: (1) is true by Proposition 3.6.

(2) For a left ideal I of R, consider the exact sequence $0 \to I \to R \to R/I \to 0$. Then (2) follows from (1).

Theorem 3.8 *Let R be a left coherent ring. Then the following are equivalent:*

(1) $lfiD(R)$

(2) $sup\{fid(M) : M$ is any left R-module$\}$

(3) $sup\{pd(F) : F$ is an FP-injective left R-module$\}$

(4) $sup\{fid(F) : F$ is an FP-injective left R-module$\}$

Proof: (1) $\Leftrightarrow$ (2) and (4) $\leq$ (2) are obvious.

(2) $\leq$ (3) Assume $sup\{pd(F) : F$ is an FP-injective left R-module$\} = M < \infty$. Let M be any left R-module and N any FP-injective left R-module. Since $pd(N) \leq M$, it follows that $Ext_R^{m+1}(N, M) = 0$. Hence $fid(M) \leq m$.

(2) $\leq$ (4) We may assume that $sup\{fid(F) : F$ is an FP-injective left R-module$\} = n < \infty$. Let M be any left R-module. By Theorem 2.3, there is a short exact sequence $0 \to K \to F \to M \to 0$, where F is FP-injective and K is FI-injective. Thus $fid(M) \leq fid(F) \leq n$, as desired.

(3) $\leq$ (4) is easy.

Corollary 3.9 *Let R be a left coherent ring. Then the following are equivalent for an integer $n \geq 0$:*

(1) $lfiD(R) \leq n$.

(2) $pd(M) \leq n$ for all FP-injective left R-modules M.

(3) $fid(M) \leq n$ for all FP-injective left R-modules M.

(4) $pd(M) \leq n$ for all left R-modules M that are both FI-injective and FP-injective and $lfiD(R) < \infty$.

(5) $fid(M) \leq n$ for all projective left R-modules M, and $lfiD(R) < \infty$.

(6) $Ext_R^{n+j}(N, M) = 0$ for all FP-injective left R-modules M, N and $j \geq 1$.

Proof: By Theorem 3.8, it suffices to show that (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (3).

(4) $\Rightarrow$ (2) Let M be any FP-injective left R-module. Since $lfiD(R) < \infty$, $fid(M) = m$ for a nonnegative integer m by Theorem 3.8 (4). Note that every left R-module has a special $\mathcal{FI}$-preenvelope by Theorem 2.3, then there exists an exact sequence $0 \to M \to E^0 \to E^1 \to \cdots \to E^{m-1} \to E^m \to 0$, where each E^i is both FI-injective and FP-injective since $pd(E^i) \leq n$ by (4), $pd(M) \leq n$.

(5) $\Rightarrow$ (3) Let M be any FP-injective left R-module. Since $lfiD(R) < \infty, pd(M) = m$ for all integer $m \geq 0$ by Theorem 3.8 (3). Hence M admits an projective resolution $0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0$. Note that $fid(P_i) \leq n$ for each P_i by (5), so $fid(M) \leq n$ by Proposition 3.6.

Proposition 3.10 *Let R be a left coherent ring. Then $lfiD(R) \geq sup\{pd(M) :$ M is a left R-module with $id(M) < \infty\} \geq sup\{pd(M) : M$ is an injective left R-module$\}$.*

Proof: Suppose $lfiD(R) = n$. Let M be a left R-module with $id(M) = m < \infty$. Then we have an exact sequence $0 \to M \to E^0 \to E^1 \to \cdots \to E^{m-1} \to E^m \to 0$. Note that every E^i is FP-injective, so $pd(E^i) \leq n$ by Theorem 3.8, whence $pd(M) \leq n$, as desired. The second inequality is trivial.

References

[1] S.T. Aldrich, E.E. Enochs, O.M.G. Jenda and L. Oyonarte, Envelopes and covers by modules of finite injective and projective dimensions, J. Algebra, 242 (2001), 447-459.

[2] F.W. Anderson, K.R. Fuller, Rings and Categories of Modules, Spring-Verlag, New York, 1974.

[3] P.C. Eklof, J. Trlifaj, How to make Ext vanish, Bull. London. Math. Soc, 33 (1) (2001), 41-51.

[4] E.E. Enochs, O.M.G. Jenda, Relative Homological Algebra, Walter de Gruyter, Berlin, New York, 2000.

[5] D.J. Fieldhouse, Character modules, dimension, and purity, Glasgow. Math. J, 13 (1972), 144-146.

[6] S. Glaz, Commutative Coherent Rings, Lecture Notes in Math, 1371, Spinger-Verlag, Berlin, 1989.

[7] O.M.G. Jenda, The dual of the grade of a module, Arch. Math, 51 (1988), 297-302.

[8] L.X. Mao, N.Q. Ding, FI-injective and FI-flat modules, J. Algebra, 309 (2007), 367-385.

[9] B. Stenström, Coherent rings and FP-injective modules, J. London Math. Soc, 2 (1970), 323-329.

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