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On the Number of Fuzzy Subgroups of Finite Abelian $p-$Groups

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Abstract

In this paper, we determine a recurrence relation of the number of fuzzy subgroups of finite abelian $p-$groups with type $(p^n,\ p^m)$, and use the recurrence relation to obtain some explicit formulas of it for some fixed m, where p is a prime integer, and $n,\ m$ are two non-negative integers.

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1 Introduction

One of the most important problems of fuzzy group theory is to classify the fuzzy subgroups of a finite abelian group. This topic has enjoyed a rapid development in the last few years. Several papers have treated the finite cyclic groups. Such as, in [1] the number of distinct fuzzy subgroups of a finite cyclic group of square-free order is determined, while [2] deals with this number for cyclic groups of order p^nq^m, and [3] gives the number of fuzzy subgroups of the finite cyclic groups and some important steps in counting the number of fuzzy subgroups of a finite elementary abelian $p-$groups. More importantly, in paper [3] a general method of a recurrence relation is established. In the present paper, we use this method to determine a recurrence relation of the number of fuzzy subgroups of finite abelian $p-$groups with type (p^n, p^m), and obtain some explicit formulas of it for some fixed non-negative integer m.

we present some basic notions and results of fuzzy subgroup theory (for details, see [4]).

2 Preliminary Notes

Let $(G, \cdot, e)$ be a group and $\mu : G \to [0, 1]$ be a fuzzy subset of G. we say that μ is a fuzzy subgroup of G if it satisfies the next two conditions:
(a) $\mu(x, y) \geq min\{\mu(x), \mu(y)\}$, for all $x, y \in G$;
(b) $\mu(x^{-1}) \geq \mu(x)$, for all $x \in G$.
So we can easily get $\mu(x^{-1}) = \mu(x)$, for any $x \in G$, and

$\mu(e) = max\{\mu(x)|x \in G\}$.

For each $\alpha \in [0, 1]$, we define the level subset : ${}_{\mu}G_{\alpha} = \{x \in G|\mu(x) \geq \alpha\}$. These subsets allow us to characterize the fuzzy subgroups of G, in the following manner: μ is a fuzzy subgroup of G if and only if its level subsets are subgroups of G.

The fuzzy subgroups of G can be classified up to some natural equivalence relations on the set consisting of all fuzzy subsets of G. One of these(used in [3], too)is defined by

$\mu \sim \eta$ iff $(\mu(x) > \mu(y) \Leftrightarrow \eta(x) > \eta(y)$, for $all\ x, y \in G)$

and two fuzzy subgroups $\mu,\ \eta$ of G will be called distinct if $\mu \nsim \eta$.

This equivalence relation is closely connected to the concept of level subgroup. In this way, suppose that G is finite and let $\mu : G \to [0, 1]$ be a fuzzy subgroup of G. Put $\mu(G) = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ and assume that $\alpha_1 > \alpha_2 > \cdots > \alpha_r$. Then μ determines the next chain of subgroup of G which ends in G:

${}_{\mu}G_{\alpha_1} \subset {}_{\mu}G_{\alpha_2} \subset \cdots \subset {}_{\mu}G_{\alpha_r} = G$.

Moreover, for any $x \in G$ and $i = \overline{1, r}$, we have

$\mu(x) = \alpha_i \Leftrightarrow \alpha_i = max\{\alpha_j|x \in {}_{\mu}G_{\alpha_j}\} \Leftrightarrow x \in {}_{\mu}G_{\alpha_i} \setminus {}_{\mu}G_{\alpha_{i-1}}$,

where by convention, we set ${}_{\mu}G_{\alpha_0} = \emptyset$. A necessary and sufficient condition for two fuzzy subgroups $\mu,\ \eta$ of G to be equivalent with respect to $\sim$ has been said above: $\alpha \sim \eta$ if and only if μ and η have the same set of level subgroups This result shows that there exists a bijection between the equivalence classes of fuzzy subgroups of G and the set of chains of subgroups of G which end in G. So, the problem of counting all distinct fuzzy subgroups of G can be translated into a combinatorial problem on the subgroup lattice $L(G)$ of G: finding the number of all chains of subgroups of G that terminate in G.

Most of our notation is standard and will usually not be repeated here. For lattice theory and group theory concepts we refer the reader to [5, 6].

3 Main Results

Let G be a finite abelian $p-$group with type $(p^n,\ p^m)$, also $G \cong Z_{p^n} \times Z_{p^m}$, where p is a prime integer, m, n are two non-negative integers, while denote the number of distinct fuzzy subgroups of G that end in G as $f(n,\ m,\ p)$. It is clearly that the function $f(n,\ m,\ p)$ is symmetric about n and m. Thus $f(n,\ m,\ p) = f(m,\ n,\ p)$ for all non-negative integral $m,\ n$. It is also obvious that for any non-negative integral n the following identity is reached $f(n,\ 0,\ p) = f(0,\ n,\ p) = 2^n$. So we assume $n \geq m$ for conveniently.

Let C be the set consisting of all chains of subgroups of G. From [7], we know that G has $p+1$ minimal subgroups, let them be $H_1,\ H_2,\ \cdots,\ H_{1+p}$. For each $r \in \{1, 2, \cdots, p+1\}$, denote by C_r the set of all chains of the lattice internal $[G/H_r] = \{H \in L(G) | H_r \subseteq H \subseteq G\}$ that end in G. Clearly, we have: $f(n,\ m,\ p) = |C| = 2|\bigcup\limits_{r=1}^{p+1} C_r|$.

Now by applying the well-known Inclusion-Exclusion Principle, one obtain that $f(n,\ m,\ p) = 2\sum\limits_{r=1}^{p+1}(-1)^{r-1} \sum\limits_{1\leq i_1<i_2<\cdots<i_r\leq p+1} |C_{i_1} \bigcap C_{i_2} \bigcap \cdots \bigcap C_{i_r}|$.

For any $1 \leq r \leq p+1$ and $1 \leq i_1 < i_2 < \cdots < i_r \leq p+1$, the set $C_{i_1} \bigcap C_{i_2} \bigcap \cdots \bigcap C_{i_r}$ consists of all chains of the lattice

$$[G/H_{i_1}] \bigcap [G/H_{i_2}] \bigcap \cdots \bigcap [G/H_{i_r}] = [G/H_{i_1} + H_{i_2} + \cdots + H_{i_r}].$$

That the interval $[G/H_{i_1} + H_{i_2} + \cdots + H_{i_r}]$ is in fact the subgroup lattice of the quotient of G with respect to $H_{i_1} + H_{i_2} + \cdots + H_{i_r}$ and this quotient maybe isomorphic to $Z_{p^{n-1}} \times Z_{p^m},\ Z_{p^n} \times Z_{p^{m-1}},\ or Z_{p^{n-1}} \times Z_{p^{m-1}}$.
then we have

$$|C_{i_1} \bigcap C_{i_2} \bigcap \cdots \bigcap C_{i_r}| = f(n-1,\ m,\ p),\ f(n, m-1, p),\ or f(n-1,\ m-1,\ p)$$

Let we prove some theorems on the base of above discussion.

Theorem 3.1 The number $f(n,\ m,\ p)$ of all distinct fuzzy subgroups of G satisfies recurrence relation:

(a) If $n \geq m \geq 1$, then

$$f(n,\ m,\ p) = 2f(n-1,\ m,\ p) + 2pf(n,\ m-1,\ p) - 2pf(n-1,\ m-1,\ p);$$

(b) If $n \geq 0,\ m = 0$, then $f(n, m, p) = 2^n$.

Proof (a) If $n = 1$, then $G = Z_p \times Z_p$ is a finite elementary abelian $p-$group with type $(p,\ p)$. We can easily get the number $f(1,\ 1,\ p)$ of fuzzy subgroups of it from [3,lemma9]: $f(1,\ 1,\ p) = 4 + 2p$. It consists with our recurrence relation.

If $n > 1$, then $G =< a,\ b | \circ(a) = p^n,\ \circ(b) = p^m,\ [a,\ b] = 1 >\cong Z_p^n \times Z_p^m$. Assume that: $H_1 =< a^{p^{n-1}} >,\ H_2 =< a^{p^{n-1}} b^{p^{m-1}} >,\ \cdots,\ H_{p+1} =< b^{p^{m-1}} >$ are all minimal subgroups of G.
If $r = 1$, then $[G/H_1] \cong Z_p^{n-1} \times Z_p^m$ and $[G/H_i] \cong Z_p^n \times Z_p^{m-1}$ for $i = \overline{2,\ p+1}$.
If $2 \leq r \leq p+1$ and $1 \leq i_1 < i_2 < \cdots < i_r \leq p+1$, then
$[G/H_{i_1} + \cdots + H_{i_r}] \cong Z_p^{n-1} \times Z_p^{m-1}$. Thus we have:

$$f(n,\ m,\ p) = 2\sum_{r=1}^{p+1}(-1)^{r-1} \sum_{1\leq i_1<i_2<\cdots<i_r\leq p+1} |C_{i_1} \bigcap C_{i_2} \bigcap \cdots \bigcap C_{i_r}|$$

$= 2[f(n-1,\ m,\ p) + C(p,\ 1)f(n,\ m-1,\ p)] \quad (r = 1)$
$-2C(p+1,\ 2)f(n-1,\ m-1,\ p) \quad (r = 2)$
$+ \cdots +$
$+2(-1)^{r-1}C(p+1,\ r)f(n-1,\ m-1,\ p) \quad (r)$
$+ \cdots +$

$+2(-1)^p C(p+1,\ p+1)f(n-1,\ m-1,\ p) \quad (r = p+1).$
$= 2[f(n-1,\ m,\ p) + pf(n,\ m-1,\ p)] + 2[-C(p+1,\ 2) + \cdots$
$+ (-1)^{r-1}C(p+1,\ r) + 2(-1)^p C(p+1,\ p+1)]f(n-1,\ m-1,\ p).$
Now, for any x, $(1+x)^{p+1} = C(p+1,\ 0)+C(p+1,\ 1)x+\cdots+C(p+1,\ p+1)x^{p+1}$. So if $x = -1$, then
$-C(p+1,\ 0) + C(p+1,\ 1) - C(p+1,\ 1) + \cdots + C(p+1,\ p+1)(-1)^p = 0.$
We get $-C(p+1,\ 1)+\cdots+C(p+1,\ p+1)(-1)^p = C(p+1,\ 0)-C(p+1,\ 1) = -p$.
Thus, $f(n,\ m,\ p) = 2f(n-1,\ m,\ p)+2pf(n,\ m-1,\ p)]-2pf(n-1,\ m-1,\ p)$.

(b) If $m = 0$, then G is a finite cyclic group. Hence $f(n,\ 0,\ p) = 2^n$ is trivial.

This recurrence relation (a) is not simple enough. Next we can get it simpler.

corollary 3.2 $f(n,\ m,\ p) = 2f(n-1,\ m,\ p) + \sum_{i=1}^{m} p^i f(n, m-i, p)$, $n \geq m \geq 1$.

Proof From the recurrence relation of theorem 3.1, we get $f(n,\ m,\ p) - 2f(n-1,\ m,\ p) = pf(n,\ m-1,\ p) + p[f(n,\ m-1,\ p) - 2f(n-1,\ m-1,\ p)]$. Assume that $g(n,\ m,\ p) = f(n,\ m,\ p) - 2f(n-1,\ m,\ p)$, then $g(n,\ 0,\ p) = 0$ and
$g(n,\ m,\ p) = pf(n,\ m-1,\ p) + pg(n,\ m-1,\ p)$
$= pf(n,\ m-1,\ p) + p[pf(n,\ m-2,\ p) + pg(n,\ m-2,\ p)]$
$= pf(n,\ m-1,\ p) + p^2 f(n,\ m-2,\ p) + p^2 g(n,\ m-2,\ p)$
$= \cdots$
$= pf(n,\ m-1,\ p) + p^2 f(n,\ m-2,\ p) + \cdots + p^m f(n,\ 0,\ p) + p^m g(n,\ 0,\ p)$
$= \sum_{i=1}^{m} p^i f(n,\ m-i,\ p).$
So we have $f(n,\ m,\ p) = 2f(n-1,\ m,\ p) + \sum_{i=1}^{m} p^i f(n,\ m-i,\ p)$.

Following the this way, we can calculate the number of fuzzy subgroups of $G = Z_{p^n} \times Z_{p^m}$ for the fixed positive integer $m,\ n$.

Remark3.3 In the calculating process, if $m > n$, we must $f(m,\ n,\ p)$ instead of $f(n,\ m,\ p)$. For example, $f(3,\ 2,\ p)$ can instead of $f(2,\ 3,\ p)$.

Next, let us compute the number $f(n,\ m,\ p)$ of fuzzy subgroups of $G = Z_{p^n} \times Z_{p^m}$ for any non-negative integer n and fixed $m = 1,\ 2,\ 3$.

Theorem 3.4 The number $f(n,\ 1,\ p)$ of fuzzy subgroups of $G = Z_{p^n} \times Z_p$ is given by the equality : $f(n,\ 1,\ p) = 2^{n+1} + n2^n p$.
where n is a non-negative integer.

Proof From corollary 3.2, for $n \geq 1$, we know
$f(n,\ 1,\ p) = 2f(n-1,\ 1,\ p) + pf(n,\ 0,\ p) = 2f(n-1,\ 1,\ p) + 2^n p.$
With this recurrence relation, we have
$f(n,\ 1,\ p) = 2[2f(n-2,\ 1,\ p) + 2^{n-1}p] + 2^n p$
$= \cdots$

$= 2^{n+1} + n \cdot 2^n p$.
If $n = 0$, then $f(0,\ 1,\ p) = f(1,\ 0,\ p) = 2 = 2^{0+1} + 0n \cdot 2^0 p$.
Thus, we get an explicit formula for the number $f(n,\ 1,\ p)$ of fuzzy subgroups of $G = Z_{p^n} \times Z_p,\ n \geq 0$.

In the particular case, when $p = 2$, we obtain the following result.

Corollary 3.5 The number $f(n,\ 1,\ 2)$ of fuzzy subgroups of $G = Z_{2^n} \times Z_2,\ n \geq 0$ is $f(n,\ 1,\ 2) = 2^{n+1}(n+1)$.

Theorem 3.6 The number $f(n,\ 2,\ p)$ of fuzzy subgroups of $G = Z_{p^n} \times Z_{p^2}$ is given by the equality :
$f(n,\ 2,\ p) = 2^{n+2} + (n+1)2^{n+1}p + (n-1)(n+4)2^{n-1}p^2$.
where n is a positive integer.

Proof From corollary 3.2, for $n \geq 2$, we know
$f(n,\ 2,\ p) = 2f(n-1,\ 2,\ p) + pf(n,\ 1,\ p) + p^2 f(n,\ 0,\ p)$
$= 2f(n-1,\ 2,\ p) + 2^{n+1}p + n \cdot 2^n p^2$.
With this recurrence relation, we have
$f(n,\ 2,\ p) = 2[2f(n-2,\ 2,\ p) + 2^n p + n \cdot 2^{n-1}p^2] + 2^{n+1}p + n \cdot 2^n p^2$
$= \cdots$
$= 2^{n+2} + (n+1) \cdot 2^{n+1}p + (n-1)(n+4)2^{n-1}p^2$.
If $n = 1$, then $f(1,\ 2,\ p) = f(2,\ 1,\ p) = 8 + 8p$
$= 2^{2+1} + (1+1) \cdot 2^(1+1)p + (1-1)(n+4)2^{1-1}p^2$.
Thus, we get an explicit formula for the number $f(n,\ 2,\ p)$ of fuzzy subgroups of $G = Z_{p^n} \times Z_{p^2},\ n \geq 1$.

Remark 3.7 If $n = 0$, then the formula of $f(n,\ 2,\ p)$ is wrong. This because $f(0,\ 2,\ p) = f(2,\ 0,\ p) = 4 \neq 2^{0+2} + (0+1) \cdot 2^{0+1}p + (0-1)(0+4)2^{0-1}p^2 = 4 + 2p - 2p^2$.

Especially, when $p = 2$, we obtain the following result.

Corollary 3.8 The number $f(n,\ 2,\ 2)$ of fuzzy subgroups of $G = Z_{2^n} \times Z_4,\ n \geq 1$ is $f(n,\ 2,\ 2) = 2^{n+1}n(n+5)$.

Theorem 3.9 The number $f(n,3,p)$ of fuzzy subgroups of $G = Z_{p^n} \times Z_{p^3}$ is given by the equality :
$f(n,\ 3,\ p) = 2^{n+3} + (n+2)2^{n+2}p + (n^2+5n)2^n p^2 + [\frac{n(n-1)(n-2)}{6} + 2n^2 - 8]2^n p^3$.
where n is a positive integer more than 2 .

Proof From corollary3.2, for $n \geq 3$, we know
$f(n,\ 3,\ p) = 2f(n-1,\ 3,\ p) + pf(n,\ 2,\ p) + p^2 f(n,\ 1,\ p) + p^3 f(n,\ 0,\ p)$
$= 2f(n-1,\ 3,\ p) + p(2^{n+2} + (n+1)2^{n+1}p + (n-1)(n+4)2^{n-1}p^2)$
$+ p^2(2^{n+1} + n \cdot 2^n p) + 2^n p^3)$
$= 2f(n-1,\ 3,\ p) + 2^{n+2}p + (n+2) \cdot 2^{n+1}p^2 + (n^2+5n-3) \cdot 2^{n-1}p^3$.
With this recurrence relation, we have
$f(n,\ 3,\ p) = 2[2f(n-2,\ 3,\ p) + 2^{n+1}p + (n+1) \cdot 2^n p^2 + (n-1)^2 2^{n-2}p^3$
$+ 5(n-1)2^{n-2}p^3 - 3 \cdot 2^{n-2}p^3] + 2^{n+2}p + (n+2) \cdot 2^{n+1}p^2 + (n^2+5n-3) \cdot 2^{n-1}p^3$
$= \cdots$
$= 2^{n+3} + (n+2)2^{n+2}p + (n^2+5n)2^n p^2 + [\frac{n(n-1)(n-2)}{6} + 2n^2 - 8]2^n p^3$.

If $n = 2$, then $f(2,\ 3,\ p) = f(3,\ 2,\ p) = 32 + 64p + 56p^2$
$= 2^{2+3} + (2+2)2^{2+2}p + (2^2 + 5\cdot 2)2^2p^2 + [\frac{2(2-1)(2-2)}{6} + 22^2 - 8]2^2p^3$.
Thus, we get an explicit formula for the number $f(n,\ 3,\ p)$ of fuzzy subgroups of $G = Z_{p^n} \times Z_{p^3},\ n \geq 2$.

Remark 3.10 If $n = 0,\ 1,\ 2$, then the formula of $f(n,\ 3,\ p)$ is wrong. This is easily verified.

In the particular case, when $p = 2$, we obtain the following result.

Corollary 3.11 The number $f(n,\ 3,\ 2)$ of fuzzy subgroups of $G = Z_{2^n} \times Z_{2^2},\ n \geq 2$ is $f(n,3,2) = 2^{n+2}[\frac{n(n-1)(n-2)}{3} + 5n^2 + 7n - 10)]$.

From corollary 3.2 and proving process of theorem 3.4, theorem 3.6, and theorem 3.9, we know that counting $f(n,\ 1,\ p)$ bases on $f(n,\ 0,\ p)$; counting $f(n,\ 2,\ p)$ bases on $f(n,\ 0,\ p)$ and $f(n,\ 1,\ p)$; counting $f(n,\ 3,\ p)$ bases on $f(n,\ 0,\ p), f(n,\ 1,\ p)$ and $f(n,\ 2,\ p)$; $\cdots$. In this way, we can get an explicit formula of the number $f(m,\ n,\ p)$ of fuzzy subgroups of $G = Z_{p^n} \times Z_{p^m}$ for any non-negative integer n and fixed non-negative integer m.

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