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2-Absorbing Ideals in Semirings

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Abstract

Suppose that R is a commutative semiring with identity $1 \neq 0$. In this paper, we define 2-absorbing ideals in R which is a generalization of prime ideals and prove that subtractive 2-absorbing ideals containing a Q-ideal I of R are precisely the subtractive 2-absorbing ideals of the quotient semiring R/I_Q (the difference semiring $R - I$). Also, we prove that if M is monic, 2-absorbing ideal in a polynomial semiring $R[x]$, then each coefficient ideal M_i is a 2-absorbing ideal in R but the converse is not true.

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1. Introduction

A. Badawi [3], introduced the concept of 2-absorbing ideals in a commutative ring with identity element, which is a generalization of prime ideals. In this paper we define 2-absorbing ideals in commutative semirings with identity element and characterize subtractive 2-absorbing ideals containing a Q-ideal I of a semiring R and subtractive 2-absorbing ideals of a quotient semiring $R/I_{(Q)}$ (difference semiring $R - I$). Also, we prove that if M is a monic, 2-absorbing ideal in a polynomial semiring $R[x]$, then each coefficient ideal M_i is a 2-absorbing ideal in R but the converse is not true. For the sake of completeness, we state some definitions and notations used throughout.

Definition 1.1. *A non empty set R together with two associative binary operations addition and multiplication is called a semiring if i) addition is a commutative operation ii) there exists $0 \in R$ such that $x + 0 = x = 0 + x$, $x \cdot 0 = 0 = 0 \cdot x$ for each $x \in R$ and iii) multiplication distributes over addition both from left and right.*

The concepts of commutative semiring, semiring with identity element, ideal, prime ideal, finitely generated ideal can be defined on the similar lines as in rings.

Throughout this paper we use the following notations

$$\begin{aligned}
R &: commutative \quad semiring \quad with \quad identity \quad element.\\
\mathbb{Z}_0^+(\mathbb{N}) &: the \quad set \quad of \quad non \quad negative \quad (positive) \quad integers.\\
(\mathbb{Z}_0^+, +, \cdot) &: the \quad semiring \quad of \quad all \quad non \quad negative \quad integers\\
& \quad under \quad usual \quad addition \quad and \quad multiplication.\\
(\mathbb{Z}_0^+ \cup \{\infty\}, \oplus, \odot) &: the \quad semiring \quad with \quad identity \quad element \quad \infty\\
& \quad where \quad a \oplus b = max\{a, b\} \quad and \quad a \odot b = min\{a, b\}.\\
< a > &: the \quad principal \quad ideal \quad generated \quad by \quad a.\\
< a, b > &: the \quad ideal \quad generated \quad by \quad a \quad and \quad b.\\
R[x] &: the \quad polynomial \quad semiring \quad over \quad a \quad semiring \quad R.
\end{aligned}$$

Definition 1.2. *An ideal I of a semiring R is called subtractive (= k-ideal) if $a, a + b \in I$, $b \in R$, then $b \in I$.*

Theorem 1.3. *([10], Theorem 2.1) A non-zero ideal I of $(\mathbb{Z}_0^+, +, \cdot)$ is prime if and only if $I = < p >$ for some prime number p or $I = < 2, 3 >$.*

Definition 1.4. *A proper ideal I of a semiring R is called 2-absorbing ideal if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.*

Clearly, every prime ideal of a semiring R is a 2-absorbing ideal.

Example 1.5. *In the semiring $(\mathbb{Z}_0^+, +, \cdot)$, let $I = < 4, 5 > = \{0, 4, 5, 8, 9, 10, 12, 13, 14, ...\} = \mathbb{Z}_0^+ - \{1, 2, 3, 6, 7, 11\}$. Then I is a 2-absorbing ideal but by Theorem 1.3, I is not a prime ideal.*

Theorem 1.6. *Every ideal of $R = (\mathbb{Z}_0^+ \cup \{\infty\}, \oplus, \odot)$ is 2-absorbing.*

Proof. I is an ideal of R if and only if $I = \{0, 1, 2, ..., t\} = I_t$ for some $t \in \mathbb{Z}_0^+$ or $I = \mathbb{Z}_0^+$ or $I = R$ ([6], Theorem 5). If $a \odot b \odot c \in I$, then a or b or $c = min\{a, b, c\} = a \odot b \odot c \in I$. Hence $a \odot b$ or $a \odot c$ or $b \odot c \in I$. □

If I is an ideal of a semiring R, then the radical of I is

$$\sqrt{I} = \{a \in R : a^n \in R \quad for \quad some \quad n \in \mathbb{N}\}$$

Theorem 1.7. *If I is a 2-absorbing ideal of a semiring R, then $\sqrt{I}$ is a 2-absorbing ideal of R.*

Proof. Let I be a 2-absorbing ideal of R. If $x \in \sqrt{I}$, then $x^n \in I$ for some $n \in \mathbb{N}$. Since I is a 2-absorbing ideal, $x^2 \in I$. Let $a, b, c \in R$ and $abc \in \sqrt{I}$. Then $a^2b^2c^2 = (abc)^2 \in I$. Since I is a 2-absorbing ideal, we may assume that $(ab)^2 = a^2b^2 \in I$. Now $ab \in \sqrt{I}$. Hence $\sqrt{I}$ is a 2-absorbing ideal. □

Converse of the Theorem 1.7 is not true.

Example 1.8. *In the semiring* $(\mathbb{Z}_0^+, +, \cdot)$*, let* $I = <3, 5> = \{0, 3, 5, 6, 8, 9, 10, ...\} = \mathbb{Z}_0^+ - \{1, 2, 4, 7\}$*. Then* I *is not a* 2*-absorbing ideal as* $2 \cdot 2 \cdot 2 = 8 \in I$ *but* $2 \cdot 2 \notin I$*. Clearly* $\sqrt{I} = \mathbb{Z}_0^+ - \{1\} = <2, 3>$*. By Theorem 1.3, I is a prime ideal in* $\mathbb{Z}_0^+$ *and hence it is a* 2*-absorbing ideal.*

2. 2-absorbing ideals in quotient semirings

Definition 2.1. *([1]) An ideal* I *of a semring* R *is called a* Q*-ideal if there exists a subset* Q *of* R *such that*

1) $R = \cup\{q + I : q \in Q\}$.
2) *if* q_1, $q_2 \in Q$, *then* $(q_1 + I) \cap (q_2 + I) \neq \emptyset \Leftrightarrow q_1 = q_2$.

Let I be a Q-ideal of a semiring R. Then $R/I_{(Q)} = \{q + I : q \in Q\}$ forms a semiring under the following addition "$\oplus$"and multiplication "$\odot$", $(q_1 + I) \oplus (q_2 + I) = q_3 + I$ where $q_3 \in Q$ is unique such that $q_1 + q_2 + I \subseteq q_3 + I$ and $(q_1 + I) \odot (q_2 + I) = q_4 + I$ where $q_4 \in Q$ is unique such that $q_1q_2 + I \subseteq q_4 + I$. This semiring $R/I_{(Q)}$ is called a quotient semiring of R by I and denoted by $(R/I_{(Q)}, \oplus, \odot)$ or just $R/I_{(Q)}$.

Lemma 2.2. *Let* I *be a* Q*-ideal of a semring* R *and* $a, b \in R$*. Let* $q, q', q'', q''' \in Q$ *be unique elements such that* $a+I \subseteq q+I$, $b+I \subseteq q'+I$, $q+q'+I \subseteq q''+I$, $qq' + I \subseteq q''' + I$*. Then* $a + b \in q'' + I$ *and* $ab \in q''' + I$*.*

Proof. Trivial. □

Lemma 2.3. *([2], Lemma 2.1) Let* R *be a semring,* I *a* Q*-ideal of* R *and* A *a subtractive ideal of* R *with* $I \subseteq A$*. Then* I *is* $Q \cap A$*-ideal of* A*.*

Theorem 2.4. *([2], Proposition 2.2 and Theorem 2.3) Let* R *be a semring,* I *a* Q*-ideal of* R*. Then* A *is a subtractive ideal of* R *with* $I \subseteq A$ *if and only if* $A/I_{(Q\cap A)}$ *is a subtractive ideal in* $R/I_{(Q)}$*.*

Theorem 2.5. *Let* I *be a* Q*-ideal in a semiring* R *and* A *be an ideal of* R *such that* $I \subseteq A$*. Then* A *is a subtractive* 2*-absorbing ideal if and only if* $A/I_{(Q\cap A)}$ *is a subtractive* 2*-absorbing ideal in* $R/I_{(Q)}$*.*

Proof. Let A be a subtractive 2-absorbing ideal of R and $q_1 + I, q_2 + I, q_3 + I \in R/I_{(Q)}$ be such that $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q + I \in A/I_{(Q\cap A)}$ where $q \in Q \cap A$ is a unique element such that $q_1q_2q_3 + I \subseteq q + I$. Therefore $q_1q_2q_3 = q + i \in A$ for some $i \in I \subseteq A$. Since A is a 2-absorbing ideal, we may assume that $q_1q_2 \in A$. Let $(q_1 + I) \odot (q_2 + I) = q_4 + I$ where $q_4 \in Q$ is a unique element such that $q_1q_2 + I \subseteq q_4 + I$. Hence $q_1q_2 = q_4 + i_1$ for some $i_1 \in I \subseteq A$. Since A is subtractive and $q_1q_2 \in A$, $q_4 \in A$ and hence $q_4 \in Q \cap A$. So $(q_1 + I) \odot (q_2 + I) = q_4 + I \in A/I_{(Q\cap A)}$. Hence $A/I_{(Q\cap A)}$ is a 2-absorbing ideal in $R/I_{(Q)}$. Also by Theorem 2.5, $A/I_{(Q\cap A)}$ is a subtractive ideal in $R/I_{(Q)}$. Conversely, assume that $A/I_{(Q\cap A)}$ is a subtractive 2-absorbing ideal in $R/I_{(Q)}$. By Theorem 2.5, A is a subtractive ideal. Let $a_1a_2a_3 \in A$

where $a_1, a_2, a_3 \in R$. Choose unique $q_1, q_2, q_3, q \in Q$ such that $a_1 + I \subseteq q_1 + I, a_2 + I \subseteq q_2 + I, a_3 + I \subseteq q_3 + I$ and $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q + I$ where $q_1q_2q_3 + I \subseteq q + I$. By Lemma 2.2, $a_1a_2a_3 = q + i$ for some $i \in I \subseteq A$. Since A is subtractive and $a_1a_2a_3 \in A$, $q \in A$. Now $q \in Q \cap A$. Hence $(q_1+I) \odot (q_2+I) \odot (q_3+I) = q+I \in A/I_{(Q\cap A)}$. Since $A/I_{(Q\cap A)}$ is a 2-absorbing ideal, we may assume that $(q_1 + I) \odot (q_2 + I) \in A/I_{(Q\cap A)}$. Let $q' \in Q \cap A$ be a unique element such that $(q_1 + I) \odot (q_2 + I) = q' + I$ where $q_1q_2 + I \subseteq q' + I$. By Lemma 2.2, there exists $i' \in I \subseteq A$ such that $ab = q' + i' \in A$. Hence A is a 2-absorbing ideal. □

Definition 2.6. *Let I be an ideal of a semiring R and $a, b \in R$. Define $a \sim b$ if there exists $x, y \in I$ such that $a + x = b + y$. Then $\sim$ is an equivalence relation on R. Let $[a]_I^R = \{x \in R : x \sim a\}$ be the equivalence class of $a \in R$. we shall also write this as $[a]$. Let $R - I = \{[a] : a \in R\}$. Then $R - I$ forms a semiring under addition and multiplication. it is called the difference semiring (or Bourne factor semiring) of R by I ([8]).*

A semiring homomorphism $f : R \to R'$ is called an isomorphism if f is one - one and onto.

Theorem 2.7. *([8], Theorem 2.4) Let I be a Q-ideal in a semiring R. Then the difference semiring $R - I$ and the quotient semiring $R/I_{(Q)}$ are isomorphic.*

Corollary 2.8. *Let I be a Q-ideal in a semiring R and A a subtractive ideal of $I \subseteq A$. Then A is a subtractive 2-absorbing ideal in R if and only if $A - I$ is a subtractive 2-absorbing ideal in $R - I$.*

Proof. By Lemma 2.3 and Theorem 2.7 we have, I is a Q-ideal in R and A is a subtractive ideal of R with $I \subseteq A$ implies I is $Q \cap A$-ideal in A and $A - I \cong A/I_{(Q\cap A)}$. Thus by Theorem 2.5, A is a subtractive 2-absorbing ideal in $R \Leftrightarrow A/I_{(Q\cap A)}$ is a subtractive 2-absorbing ideal in $R/I_{(Q)} \Leftrightarrow A - I$ is a subtractive 2-absorbing ideal in $R - I$. □

3. 2-absorbing ideals in polynomial semirings

L. Dale [5] has proved that if A is an ideal in $R[x]$ and $A_i = \{a \in R : \textit{there exists } f(x) \in A \textit{ such that } ax^i \textit{ is a term of } f(x)\}$, then $\{A_n\}$ is an ascending chain of ideals in R ([5], Theorem 2.2). The ideals $\{A_n\}$ are called coefficient ideals. L. Dale ([5], Theorem 3.4) has also proved that a monic ideal M in $R[x]$ is a subtractive ideal if and only if each M_i is a subtractive ideal in R. J. N. Chaudhari and V. Gupta ([4], Theorem 3.1) have proved that if a monic ideal M is a prime ideal in $R[x]$, then each M_i is a prime ideal of R, but the converse is not true. In this section we prove the following theorem.

Theorem 3.1. *Let M be a monic, 2-absorbing ideal in a polynomial semiring $R[x]$. Then each coefficient ideal M_i is a 2-absorbing ideal in R.*

Proof. Let M be a monic, 2-absorbing ideal of $R[x]$ and $abc \in M_i$. Then there exists $f(x) \in M$ such that $(abc)x^i$ is a term of $f(x)$. Since M is monic $(abc)x^i = a \cdot b \cdot cx^i \in M$. Since M is a 2-absorbing ideal, $ab \in M$ or $acx^i \in M$ or $bcx^i \in M$. Hence $ab \in M_0 \subseteq M_i$ or $ac \in M_i$ or $bc \in M_i$. Thus each coefficient ideal M_i is a 2-absorbing ideal. $\square$

Converse of the Theorem 3.1 is not true.

Example 3.2. *Let $R = (\mathbb{Z}_0^+ \cup \{\infty\}, \oplus, \odot)$ be a commutative semiring with identity ∞. Define $M = \{f(x)x^3 \in R[x] : f(x) = \sum_{i=0}^{n} a_i x^i \quad where \quad a_i \leq i+1 \quad for \quad all \quad i\}$. Then we will show that M is a monic ideal and each coefficient ideal M_i is a 2-absorbing ideal but M is not a 2-absorbing ideal.*

1) *M is an ideal : Let $f(x) = \sum_{i=0}^{n} a_i x^i x^3$, $g(x) = \sum_{i=0}^{n} b_i x^i x^3 \in M$ where $a_i, b_i \leq i+1$. Therefore $f(x) + g(x) = \sum_{i=0}^{n} (a_i \oplus b_i) x^i x^3 \in M$, since $a_i \oplus b_i = max\{a_i, b_i\} \leq i+1$. If $h(x) = \sum_{j=0}^{m} c_j x^j \in R[x]$, then $f(x)h(x) = (\sum_{i=0}^{n} a_i x^i x^3)(\sum_{j=0}^{m} c_j x^j) = \sum_{k=0}^{n+m} d_k x^k x^3$ where $d_k = \oplus \sum_{k=i+j} a_i \odot c_j$ where $a_i \odot c_j = min\{a_i, c_j\} \leq a_i \leq i+1 \leq k+1$. Hence $d_k \leq k+1$ for all k. Now $f(x)h(x) \in M$.*
2) *M is a monic ideal : Let $f(x) = \sum_{i=0}^{n} a_i x^i x^3 \in M$. Then each term $a_i x^{i+3} = a_i x^i x^3 \in M$, since $a_i \leq i+1$ for all i.*
3) *Each M_t is a 2-absorbing ideal : Clearly $M_0 = M_1 = M_2 = \{0\}$. For $t \geq 3$,*

$$
\begin{aligned}
M_t &= \{a \in R : there \quad exists \quad g(x) = \sum_{i=0}^{n} a_i x^i x^3 \in M \quad such \quad that \quad ax^t \\
&\qquad is \quad a \quad term \quad of \quad g(x)\} \\
&= \{a \in R : there \quad exists \quad g(x) = \sum_{i=0}^{n} a_i x^{i+3} \in M \quad such \quad that \\
&\qquad ax^t = a_{t-3}x^t \quad is \quad a \quad term \quad of \quad g(x) \quad where \quad a = a_{t-3} \\
&\qquad \leq t-3+1 = t-2 \quad for \quad all \quad t\} \\
&= \{a \in R : a \leq t-2\} \\
&= \{0, 1, 2, ..., t-2\}
\end{aligned}
$$

By Theorem 1.6, each M_t is a 2-absorbing ideal of R.

4) *M is not a 2-absorbing ideal : $x \cdot x \cdot x = 1 \cdot x^3 \in M$ where $a_0 = 1 \leq 0+1$ and $a_0 \in R[x]$ but $x \cdot x = 1 \cdot x^2 \notin M$.*

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