International Journal of Algebra, Vol. 6, 2012, no. 6, 277 - 280

g-Congruences on po-Γ-Semigroups

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Abstract

A congruence σ on a semigroup or an ordered semigroup S is called a *g-congruence* if for any $A \in S/\sigma$ there exists one and only one element of A which is a right unit of A. Kehayopulu and Tsingelis [1] dealt with g-congruences on an ordered semigroup. In this note, we show that the results can be obtained on an ordered Γ-semigroup.

Mathematics Subject Classification: 06F05

Keywords: Ordered semigroup (po-semigroup), ordered Γ-semigroup (po-Γ-semigroup), g-congruence, semilattice congruence, congruence

1 Introduction and Preliminaries

Let S be an ordered semigroup (po-semigroup). A congruence σ on S is called a *g-congruence* if for any $A \in S/\sigma$ there exists one and only one element e of A which is a right unit of A. In [1], the authors dealt with g-congruences on S. Firstly, it was proved that if the order $\leq$ on S is a chain, σ a g-congruence, e, f right units of S and $g \in (e)_\sigma$, $h \in (f)_\sigma$, then the following are equivalent: (1) $h \leq g$. (2) $h < gf$ or ($h = gf$ and $f \leq e$). (3) $he < g$ or ($he = g$ and $f \leq e$). Let $\mathcal{R}U_\sigma$ be the set of units of $(a)_\sigma$ for all $a \in S$. Secondly, if the order $\leq$ on S is a chain, σ a g-congruence and $g, h \in S$, then the following are equivalent: (1) $h \leq g$. (2) There exist $e, f \in \mathcal{R}U_\sigma$ such that $g \in (e)_\sigma$, $h \in (f)_\sigma$

[1]This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

and $h \leq gf$ or ($h = gf$ and $f \leq e$). (3) There exist $e, f \in \mathcal{R}U_\sigma$ such that $g \in (e)_\sigma$, $h \in (f)_\sigma$, and $he < g$ or ($g = he$ and $f \leq e$). In this paper, we show that analogous results can be obtained on an ordered Γ-semigroup.

Let S and Γ be nonempty sets. Then S is called a Γ-*semigroup* [3] if there is a mapping $S \times \Gamma \times S \to S$, written as $(x, \gamma, y) \mapsto x\gamma y$, such that

$$(x\gamma y)\beta z = x\gamma(y\beta z)$$

for all $x, y, z \in S$ and all $\gamma, \beta \in \Gamma$.

Definition 1.1 *A Γ-semigroup S is called an ordered Γ-semigroup (po-Γ-semigroup) if there is a relation $\leq$ on S such that $x \leq y$ implies $x\gamma z \leq y\gamma z$ and $z\gamma x \leq z\gamma y$ for any $x, y, z \in S$ and all $\gamma \in \Gamma$.*

Let S be a Γ-semigroup or *po*-Γ-semigroup and $\alpha \in \Gamma$. An element e of S is called α-*idempotent* if $e\alpha e = e$. Let $E_\alpha(S)$ denote the set of all α-idempotents of S. Let $E(S) = \bigcup_{\alpha \in \Gamma} E_\alpha(S)$.

Let S be a Γ-semigroup or *po*-Γ-semigroup. An equivalence relation σ on S is called a *congruence* relation on S if for $x, y, z \in S$ and $\alpha \in \Gamma$, $(x, y) \in \sigma$ implies $(x\alpha z, y\alpha z) \in \sigma$ and $(z\alpha x, z\alpha y) \in \sigma$.

Definition 1.2 *An equivalence relation σ on a Γ-semigroup or po-Γ-semigroup S is called g-congruence (group-congruence) if for every $A \in S/\sigma$ there exists one and only one element e of A, called right unit of A, such that $a\alpha e = a$ for all $a \in A$ and for all $\alpha \in \Gamma$.*

By Definition 1.2, if σ is a g-congruence on a Γ-semigroup or *po*-Γ-semigroup S, then for each $A \in S/\sigma$ contains a unique right unit.

Definition 1.3 *Let S be a Γ-semigroup and σ a g-congruence on S. Let*

$$\mathcal{R}U_\sigma = \{e \in S \mid e \textit{ is a right unit of } (a)_\sigma; a \in S\}.$$

Note that $\mathcal{R}U_\sigma = \{e \in S \mid e \text{ is a right unit of } (e)_\sigma\}$.

2 Main Results

Proposition 2.1 *Let S be a Γ-semigroup and σ a g-congruence on S.*

(i) $\mathcal{R}U_\sigma \subseteq E(S)$.

(ii) *If $e, f \in \mathcal{R}U_\sigma$, then $(e)_\sigma = (f)_\sigma$ if and only if $e = f$.*

(iii) $S = \bigcup_{e \in \mathcal{R}U_\sigma} (e)_\sigma$.

Proof. The proofs are modifications of the proofs in [1].

(i) Let $e \in \mathcal{R}U_\sigma$. Then e is a right unit of $(e)_\sigma$. Since $e\alpha e = e$ for all $\alpha \in \Gamma$, we obtain $e \in E(S)$.

(ii) Let $e, f \in \mathcal{R}U_\sigma$ be such that $(e)_\sigma = (f)_\sigma$. Since e is a right unit of $(f)_\sigma$ and f is a right unit of $(f)_\sigma$ and σ is a g-congruence , it follows that $e = f$. It is clear that $e = f$ implies $(e)_\sigma = (f)_\sigma$.

(iii) Let $x \in S$. Since σ is a g-congruence, $(x)_\sigma$ contains the right unit element e. Since $(x)_\sigma = (e)_\sigma$, $x \in (e)_\sigma$. Since e is the right unit element of $(e)_\sigma$, we have $x \in \bigcup_{e \in \mathcal{R}U_\sigma}(e)_\sigma$.

Theorem 2.2 *Let S be a po-Γ-semigroup where $\leq$ is a chain, and σ a g-congruence on S. Let $e, f \in \mathcal{R}U_\sigma$ and $g \in (e)_\sigma, h \in (f)_\sigma$. The following are equivalent:*

(i) *$h \leq g$.*

(ii) *For $\alpha \in \Gamma$, $h < g\alpha f$ or ($h = g\alpha f$ and $f \leq e$).*

(iii) *For $\alpha \in \Gamma$, $h\alpha e < g$ or ($h\alpha e = g$ and $f \leq e$).*

Proof. (i) $\Rightarrow$ (ii). Assume that $h \leq g$. Let $\alpha \in \Gamma$. Then $h\alpha f \leq g\alpha f$. Since f is a right unit of $(f)_\sigma$ and $h \in (f)_\sigma$, we have $h \leq g\alpha f$. Therefore, $h < g\alpha f$ or $h = g\alpha f$. Assume that $h = g\alpha f$ and $e < f$. Since $g\alpha e \leq g\alpha f$, we obtain $g\alpha e \leq h$. Since $g \in (e)_\sigma$, it follows that $g \leq h$. Thus $g = h$, and $(h)_\sigma = (g)_\sigma$. Since $(e)_\sigma = (f)_\sigma$, by Proposition 2.1, we get $e = f$. A contradiction.

(ii) $\Rightarrow$ (i). Assume that for $\alpha \in \Gamma$, $h < g\alpha f$ or ($h = g\alpha f$ and $f \leq e$). There are two cases to consider. If $h < g\alpha f$ and $g < h$, then $g\gamma f \leq h\gamma f$ for all $\gamma \in \Gamma$. Since f is a right unit of $(f)_\sigma$, we have $g\gamma f \leq h$ for all $\gamma \in \Gamma$. Thus $g\alpha f \leq h$. A contradiction. If $h = g\alpha f$ and $f \leq e$, then $g\gamma f \leq g\gamma e$ for all $\gamma \in \Gamma$. Thus $h = g\alpha f \leq g\alpha e$. Since e is a right unit of $(e)_\sigma$, we have $h \leq g$.

That (i) $\Leftrightarrow$ (iii) can be proved similarly.

Theorem 2.3 *Let S be a po-Γ-semigroup where $\leq$ is a chain and σ a g-congruence on S. Let $e, f \in \mathcal{R}U_\sigma$, $e \neq f$ and $g \in (e)_\sigma, h \in (f)_\sigma$. The following are equivalent:*

(i) *$h \leq g$.*

(ii) *$h < g$.*

(iii) *For $\alpha \in \Gamma$, $h < g\alpha f$ or ($h = g\alpha f$ and $f \leq e$).*

(iv) *For $\alpha \in \Gamma$, $h\alpha e < g$ or ($h\alpha e = g$ and $f \leq e$).*

Proof. (i) ⇒ (ii). Assume that $h \leq g$. If $h = g$, then $(h)_\sigma = (g)_\sigma$, so $(e)_\sigma = (f)_\sigma$. By Proposition 2.1, $e = f$. A contradiction. Thus $h < g$.

(ii) ⇒ (iii). Suppose that $h < g$. Then $h \leq g$. Since $e \neq f$, by Theorem 2.2, we have (iii).

(iii) ⇒ (iv). Assume that (iii) holds. Since $f < e$, $f \leq e$. By Theorem 2.2, (iv) follows.

(iv) ⇒ (i). This follows from Theorem 2.2 (iii) ⇒ (i).

Theorem 2.4 *Let S be a po-Γ-semigroup where $\leq$ is a chain and σ a g-congruence on S. Let $g, h \in S$. The following are equivalent:*

(i) $h \leq g$.

(ii) *There exist $e, f \in \mathcal{R}U_\sigma$ such that $g \in (e)_\sigma, h \in (f)_\sigma$, and for $\alpha \in \Gamma$, $h < g\alpha f$ or ($h = g\alpha f$ and $f \leq e$).*

(iii) *There exist $e, f \in \mathcal{R}U_\sigma$ such that $g \in (e)_\sigma, h \in (f)_\sigma$, and for $\alpha \in \Gamma$, $h\alpha e < g$ or ($h\alpha e = g$ and $f \leq e$).*

Proof. Assume that $h \leq g$. Since σ is a g-congruence on S, by Proposition 2.1, we have $S = \bigcup_{e \in \mathcal{R}U_\sigma} (e)_\sigma$. Since $g, h \in S$, we obtain $g \in (e)_\sigma$ and $h \in (f)_\sigma$ for some $e, f \in \mathcal{R}U_\sigma$. By Theorem 2.2, (ii) follows. Hence (i) ⇒ (ii).

Using Theorem 2.2 (ii) ⇒ (iii) and (iii) ⇒ (i), we have (ii) ⇒ (iii) and (iii) ⇒ (i), respectively.

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Received: October, 2011