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Free Objects in the Variety of Groupoids Defined by the Identity $xf(x) \approx f(f(x))$

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Abstract

Let $\mathcal{V}_f$ denote the variety of groupoids defined by the identity $xf(x) \approx f(f(x))$, where f is a fixed nontrivial groupoid power. A description of free objects in this variety and their characterization by means of injective groupoids in $\mathcal{V}_f$ are obtained.

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1 Introduction

A construction of free objects in the variety of groupoids defined by the identity $xx^2 \approx x^2x^2$ and their characterization are presented in [4]. A generalization of this problem is investigated in [2]. The variety of groupoids defined by the identity $xf(x) \approx f(f(x))$, where f is a fixed nontrivial groupoid power is another generalization of [4]. We denote this variety by $\mathcal{V}_f$.

Throughout the paper we will use the concept of groupoid power and some of its properties stated in [5]. For the notation and basic notions of universal algebra the reader is referred to [8]. In most cases, without mention, the operation is denoted multiplicatively: the product of two elements x, y of a groupoid is denoted by $x \cdot y$ or just xy.

2 Preliminaries

Let X be an arbitrary nonempty set whose elements are called variables and T_X be the set of all groupoid terms over X in signature $\cdot$. The terms are denoted by $t, u, v, w \ldots$ The term groupoid $\boldsymbol{T}_X = (T_X, \cdot)$, where the operation is defined by $(t, u) \mapsto tu$, is an absolutely free groupoid over X. The groupoid $\boldsymbol{T}_X$ is *injective*, i.e. the operation $\cdot$ is an injective mapping: $tu = vw \Rightarrow t = v, u = w$. The set X is the set of primes in $\boldsymbol{T}_X$ that generates $\boldsymbol{T}_X$. (An element a of a groupoid $\boldsymbol{G} = (G, \cdot)$ is said to be *prime* in $\boldsymbol{G}$ if $a \neq xy$, for all $x, y \in G$.) These two properties of $\boldsymbol{T}_X$ characterize all absolutely free groupoids.

Proposition 2.1 ([1]; Lemma 1.5)*A groupoid $\boldsymbol{H} = (H, \cdot)$ is an absolutely free groupoid if and only if it satisfies the following two conditions:*
(i) $\boldsymbol{H}$ is injective.
(ii) The set of primes in $\boldsymbol{H}$ is nonempty and generates $\boldsymbol{H}$.
Then the set of primes is the unique free generating set of $\boldsymbol{H}$.

We refer to this proposition as *Bruck Theorem* for the class of all groupoids.

For any term t we define the *length* $|\,t\,|$ of t, the *set of subterms* $P(t)$ of t and a *content* $cn(t)$ of t in the following inductive way:

$$t \in X \Rightarrow |\,t\,| = 1, \quad t = t_1t_2 \Rightarrow |\,t_1t_2\,| = |\,t_1\,| + |\,t_2\,|$$
$$t \in X \Rightarrow P(t) = \{t\}, \quad t = t_1t_2 \Rightarrow P(t_1t_2) = \{t_1t_2\} \cup P(t_1) \cup P(t_2)$$
$$t \in X \Rightarrow cn(t) = \{t\}, \quad t = t_1t_2 \Rightarrow cn(t) = cn(t_1) \cup cn(t_2).$$

By $\boldsymbol{E} = (E, \cdot)$ we denote the term groupoid with one-element generating set $\{e\}$. The elements of E are called *groupoid powers* ([5]) and they will be denoted by $f, g, h, \ldots$. We say that e is the *trivial* groupoid power. A few properties of groupoid powers that will be used further on are stated below.

For any groupoid $\boldsymbol{G} = (G, \cdot)$, each element $f \in E$ induces a transformation $f^{\boldsymbol{G}} : G \to G$, called an *interpretation* of f in $\boldsymbol{G}$, defined by:

$$e^{\boldsymbol{G}}(x) = x, \;\; (gh)^{\boldsymbol{G}}(x) = g^{\boldsymbol{G}}(x)h^{\boldsymbol{G}}(x)$$

for any $g, h \in E$ and $x \in G$. We write $f(x)$ instead of $f^{\boldsymbol{G}}(x)$ when $\boldsymbol{G}$ is understood, $f(t)$ instead of $f^{\boldsymbol{T}_X}(t)$ and $f(g)$ instead of $f^{\boldsymbol{E}}(g)$.

The following statements are shown in [7].

Proposition 2.2 *If $f, g \in E$, $t, u \in T_X$, then:*
a) $|\,f(t)\,| = |\,f\,| \cdot |\,t\,|$;
b) $f(t) = g(u) \;\wedge\; (|\,t\,| = |\,u\,| \;\vee\; |\,f\,| = |\,g\,|) \;\Rightarrow\; (f = g \;\wedge\; t = u)$;
c) $f(t) = g(u) \;\wedge\; |\,t\,| \geq |\,u\,| \;\Leftrightarrow (\exists!\, h \in E)\; (t = h(u) \;\wedge\; g = f(h))$.

If an operation "$\circ$" is defined on the set E by $f \circ g = f(g)$, then one obtains that $(E, \circ, e)$ is a cancellative monoid ([5]). We obtain an algebra $(E, \circ, \cdot, e)$ with two binary operations $\circ$ and $\cdot$, and a unary operation e, such that $\circ$ is right-distributive with respect to $\cdot$, i.e. for any $g, f_1, f_2 \in E$,

$$e \circ g = g \circ e = g, \quad (f_1f_2) \circ g = (f_1 \circ g)(f_2 \circ g).$$

Given a groupoid $\boldsymbol{G}$, an element $a \in G$ is said to be *primitive* in $\boldsymbol{G}$ if $a \neq f(b)$, for any $b \in G$ and any $f \in E \setminus \{e\}$. An element of $\boldsymbol{G}$ is said to be *potent* in $\boldsymbol{G}$ if it is not primitive. Specially, primitive elements and potent elements in $\boldsymbol{T}_X$ are called *primitive terms* and *potent terms*, respectively.

The following proposition is shown in ([3]).

Proposition 2.3 1°. *For any term $t \in T_X$ there is a unique primitive term $u \in T_X$ and a unique $f \in E$ such that $t = f(u)$.*

In that case we call u the *base* of t, f the *power* of t, $|\,f\,|$ the *exponent* of t, and denote by $\underline{t} = u$, $t^{\sim} = f$ and $|\,t^{\sim}\,|$, respectively.

2°. *If $s, t \in T_X$ and $\underline{s} \neq \underline{t}$, then $\underline{st} = st$ and $(st)^{\sim} = e$.*

3°. *If $s, t \in T_X$ and $\underline{s} = \underline{t} = u$, then $\underline{st} = u$ and $(st)^{\sim} = s^{\sim}t^{\sim}$.*

3 A construction of canonical groupoids in $\mathcal{V}_f$

We say that a groupoid $\boldsymbol{R} = (R, *)$ is a *canonical groupoid* in $\mathcal{V}_f$ if it satisfies the following conditions ([6]):

(c_0) $X \subseteq R \subseteq T_X$;

(c_1) $tu \in R \Rightarrow t, u \in R \ \wedge \ t * u = tu$;

(c_2) $\boldsymbol{R}$ is a free groupoid in $\mathcal{V}_f$ over X (i.e. $\mathcal{V}_f$-free groupoid over X).

Define the carrier R of the desired groupoid $(R, *)$ by:

$$R = \{t \in T_X : \ (\forall x \in T_X) \ \ xf(x) \notin P(t)\}. \tag{1}$$

As an obvious consequence of (1) we obtain the following

Proposition 3.1 *a) $X \subseteq R \subseteq T_X$; $\ t \in R \ \Rightarrow \ P(t) \subseteq R$.*

b) $t, u \in R \Rightarrow [\,tu \notin R \Leftrightarrow u = f(t)\,]$.

c) $t, u \in T_X \Rightarrow [\,tu \in R \Leftrightarrow t, u \in R \ \wedge \ u \neq f(t)\,]$.

d) $t \in R \Rightarrow t^n \in R$, where n is a positive integer.

By induction on the length of g one can show the following proposition.

Proposition 3.2 *If $t \in R$, then $g(t) \in R$ for any $g \in P(f)$.*

Corollary 3.3 *If $t \in R$, then $f(t) \in R$ and $f(f(t)) \in R$.*

Define an operation $*$ on R by:

$$t, u \in R \ \Rightarrow t * u = \begin{cases} tu, \text{ if } \ tu \in R \\ f(f(t)), \text{ if } \ u = f(t). \end{cases} \tag{2}$$

By (2) it is clear that $\boldsymbol{R}$ is a groupoid and that the set of primes in $\boldsymbol{R}$ coincides with X. By induction on length of a term one can show that X is the generating set for $\boldsymbol{R}$. To show that $\boldsymbol{R} \in \mathcal{V}_f$ we use the fact that $g_*(t) = g(t)$, for any $t \in R$ and every $g \in P(f)$. Here $f_*(t)$ denotes the *interpretation of* $f \in E$ *in* $\boldsymbol{R}$ and it is defined by:

$$e_*(t) = t, \ \ (f_1 f_2)_*(t) = (f_1)_*(t) * (f_2)_*(t),$$

for every $t \in R$.

Put $f = f_1 f_2$. By Prop.3.2 it follows that $f_*(t) \in R$. Therefore: $f_*(f_*(t)) = f_*(f(t)) = f(f(t)) = t * f(t) = t * f_*(t)$, i.e. the identity $xf(x) \approx f(f(x))$ is satisfied in $\boldsymbol{R}$. Thus, $\boldsymbol{R} \in \mathcal{V}_f$.

The groupoid $\boldsymbol{R} = (R, *)$ has the universal mapping property in $\mathcal{V}_f$ over X. Namely, let $\boldsymbol{G} \in \mathcal{V}_f$ and $\lambda : X \to G$ be any mapping and φ be the homomorphism from $\boldsymbol{T}_X$ into $\boldsymbol{G}$, such that $\varphi|_X = \lambda$. Put $\psi = \varphi|_R$. Then ψ is a homomorphism from $\boldsymbol{R}$ into $\boldsymbol{G}$ that is an extension of λ. It suffices to show that $\varphi(t * u) = \varphi(tu)$, for any $t, u \in R$. If $tu \in R$, than $t * u = tu$ and $\varphi(t * u) = \varphi(tu)$. If $tu \notin R$, then by Prop.3.1 b) it follows that $u = f(t)$ and so $\varphi(t * u) = \varphi(t * f(t)) = \varphi(f(f(t)))$. By induction on the length of f one can show that $\varphi(f(t)) = f(\varphi(t))$. Thus, $\varphi(t * u) = f(\varphi(f(t))) = f(f(\varphi(t))) = [\boldsymbol{G} \in \mathcal{V}_f] = \varphi(t)(f(\varphi(t))) = \varphi(t)\varphi(f(t)) = \varphi(t)\varphi(u) = \varphi(tu)$.

Hence, the conditions (c_0), (c_1) and (c_2) are fulfilled and thus the following theorem holds.

Theorem 3.4 *The groupoid* $(R, *)$ *is a canonical groupoid in* $\mathcal{V}_f$ *over* X.

Note that, by a direct verification, one can show that the groupoid $(R, *)$ is left cancellative, but it is not right cancellative.

4 Injective objects in $\mathcal{V}_f$

$\mathcal{V}_f$-free groupoids can be characterized by a subclass of $\mathcal{V}_f$, called the class of $\mathcal{V}_f$-injective groupoids. This subclass is larger then the class of $\mathcal{V}_f$-free groupoids. We define the class of $\mathcal{V}_f$-injective groupoids by using the properties of the obtained canonical groupoid $\boldsymbol{R} = (R, *)$ in $\mathcal{V}_f$ concerning the non-prime elements in $\boldsymbol{R}$. This class will be successfully defined if the following conditions are fulfilled:

(i_0) Every $\mathcal{V}_f$-injective groupoid $\boldsymbol{H}$ whose set of primes is nonempty and generates $\boldsymbol{H}$ is $\mathcal{V}_f$-free.

(i_1) The class of $\mathcal{V}_f$-free groupoids is a proper subclass of the class of $\mathcal{V}_f$-injective groupoids.

For that purpose it is necessary to establish the cases when two products $t * s$ and $u * v$ are equal and to solve the equation $t * s = u * v$ in the obtained groupoid $\boldsymbol{R}$, for given $u, v \in R$. Solving this equation we obtain:

Proposition 4.1 *Let* $z \in R \setminus X$*, i.e.* $z = u * v$*, for some* $u, v \in R$.
a) *If* $uv \in R$*, then* (u, v) *is the unique pair of divisors of* z.

b) If $uv \notin R$ and $f = f_1 f_2$, then z has two pairs of divisors:
$(f_1(f(u)), f_2(f(u)))$ and $(u, f(u))$.

Before introducing the notion of $\mathcal{V}_f$-injectivity we present a few necessary notions and results.

Definition 4.2 Let $\boldsymbol{G}$ be a groupoid and $f \in E \setminus \{e\}$ be a fixed groupoid power. An element $a \in G$ is said to be *f-potent* in $\boldsymbol{G}$ if there is $b \in G$, such that $a = f(b)$. If $a \neq f(b)$ for every $b \in G$, then we say that a is an *f-primitive element* in $\boldsymbol{G}$.

For every $k \geq 0$ define a transformation $[k] : x \mapsto f^{[k]}(x)$ on $\boldsymbol{G}$ by:

$$f^{[0]}(x) = x, \;\; f^{[1]}(x) = f(x), \;\; f^{[k+1]}(x) = f^{[k]}(f(x)).$$

For instance, $f^{[2]}(x) = f(f(x))$. From the definition of the operation $\circ$ in E we obtain that $f^{[2]}(x) = (f \circ f)(x)$. Note that $f^{[p+q]}(x) = f^{[p]}(f^{[q]}(x)) = f^{[q]}(f^{[p]}(x))$, for any nonnegative integers p, q.

One can show the following proposition by induction on p or q, the associativity of $\circ$ and by the injectivity of $\boldsymbol{T}_X$.

Proposition 4.3 *If $t, u \in T_X$ and p, q are non-negative integers, then:*
a) $|f^{[p]}(t)| = |\,f\,|^p \cdot |\,t\,|$. *b) $f^{[p]}(t) = f^{[p]}(u) \;\Rightarrow\; t = u$.*
c) $f^{[p]}(t) = f^{[q]}(t) \;\Rightarrow\; p = q$. *d) $f^{[p]}(t) = f^{[p+q]}(u) \;\Rightarrow\; t = f^{[q]}(u)$.*
e) If t, u are f-primitive terms, then: $f^{[p]}(t) = f^{[q]}(u) \Rightarrow t = u,\; p = q$.

Proposition 4.4 *For every term $t \in T_X$ there is a unique f-primitive term $\alpha \in T_X$ and a unique $k \geq 0$, such that $t = f^{[k]}(\alpha)$.*

In that case we say that α is an *f-base* of t; $f^{[k]}$ is an *f-power* of t; k is an *f-exponent* of t and denote them by: $\alpha = \underline{t}$, $f^{[k]} = t^{\sim}$ and $k = |t^{\sim}|$, respectively. If t is an f-primitive term, then $\underline{t} = t$ and $|t^{\sim}| = 0$.

Proof. *Existence.* Let $t \in T_X$. If t is an f-primitive term, then $t = e(t) = f^{[0]}(t)$. Let t be an f-potent term. Then, there is a term u_1, such that $t = f(u_1)$. If u_1 is f-primitive, then $t = f^{[1]}(u_1)$. If u_1 is f-potent, then there is a term u_2 such that $u_1 = f(u_2)$. If u_2 is f-primitive, then the procedure ends and $t = f^{[1]}(f^{[1]}(u_2)) = f^{[1]}(f(u_2)) = f^{[2]}(u_2)$. Note that $|\,t\,| > |\,u_1\,| > |\,u_2\,|$. If u_2 is f-potent, then the procedure continues. Hence, a descending sequence $(|\,u_i\,|)$ of positive integers is obtained. This sequence must end since $|\,t\,|$ is a positive integer. Thus, there is a term u_n that is f-primitive and $t = f^{[n]}(u_n)$.

Uniqueness. Let $t = f^{[k]}(\alpha)$ and $t = f^{[m]}(\beta)$ for some f-primitive terms α, β and $k, m \geq 0$. If $k = m$, then $f^{[k]}(\alpha) = f^{[m]}(\beta)$. By Prop.4.3 b) it follows that $\alpha = \beta$. Let $k \neq m$. Assume that $k < m$ and that $m = k + l$, for $l \geq 1$. Then $f^{[k]}(\alpha) = f^{[k+l]}(\beta)$. By Prop.4.3 d) it follows that $\alpha = f^{[l]}(\beta)$, that contradicts the assumption that α is an f-primitive term.

Considering that $f_*(t) = f(t)$ for every $t \in R$, then we can conclude the following

Corollary 4.5 *For every $t \in R$ there is a unique f-primitive term $\alpha \in R$ and a unique $k \geq 0$, such that $t = f^{[k]}(\alpha)$.*

Proposition 4.6 *If $t, u \in T_X$ and p, q are non-negative integers, then:*

$$f^{[p]}(t) = f^{[q]}(u) \;\Rightarrow\; \underline{t} = \underline{u} \;\;\wedge\;\; p + |t^{\sim}| = q + |u^{\sim}|.$$

Proof. Let $f^{[p]}(t) = f^{[q]}(u)$. If $p = q$ and t, u are f-primitive terms, then by Prop.4.3 b), $t = u$ and thus $\underline{t} = \underline{u}$. In that case, $p + |t^{\sim}| = q + |u^{\sim}|$. If $p = q$ and t, u are f-potent terms, then by Prop.4.4 there are unique f-primitive terms t_1, u_1 and unique $n, m \geq 0$, such that $t = f^{[n]}(t_1)$, $u = f^{[m]}(u_1)$. In that case $f^{[p]}(f^{[n]}(t_1)) = f^{[q]}(f^{[m]}(u_1))$, and by Prop.4.3 e), $t_1 = u_1$, so $\underline{t_1} = \underline{u_1}$. Hence $f^{[p+n]}(t_1) = f^{[q+m]}(t_1)$, which implies that $n = m$. Obviously, $p + |t^{\sim}| = q + |u^{\sim}|$. Let $p \neq q$. Assume that $q = p + k$, where $k \geq 1$, i.e. $f^{[p]}(t) = f^{[p+k]}(u)$. By Prop.4.3 d), $t = f^{[k]}(u)$. If u is f-primitive, then $\underline{t} = u = \underline{u}$ and $|\, t^{\sim} \,| = k$, $|\, u^{\sim} \,| = 0$, so $p + |t^{\sim}| = p + k = q + 0 = q + |u^{\sim}|$. If u is f-potent, then there is a unique f-primitive term u_1 and a unique $k_1 > 0$, such that $u = f^{[k_1]}(u_1)$. Hence, $t = f^{[k]}(f^{[k_1]}(u_1)) = f^{[k+k_1]}(u_1)$. Therefore $\underline{t} = u_1 = \underline{u}$ and $|\, t^{\sim} \,| = k + k_1$, $|\, u^{\sim} \,| = k_1$. So, $p + |t^{\sim}| = p + k + k_1 = q + k_1 = q + |u^{\sim}|$.

Proposition 4.7 *Let $t \in R \setminus X$ and $f = f_1 f_2$.*
a) t is an f-primitive term in $\boldsymbol{R}$ if and only if t is an f-primitive term in $\boldsymbol{T}_X$.
b) If t is an f-primitive term in $\boldsymbol{R}$ then there is a unique pair $(u, v) \in R \times R$ such that $t = u * v$.
c) If t is an f-potent term such that $t = f(\alpha)$, where α is an f-primitive term in $\boldsymbol{R}$, then there is a unique pair $(u, v) \in R \times R$ such that $t = u * v$ and $u = f_1(\alpha)$, $v = f_2(\alpha)$.
d) If t is an f-potent term such that $t = f^{[k+2]}(\alpha)$, $k \geq 0$ and α is an f-primitive term in $\boldsymbol{R}$, then the pairs

$$(f^{[k]}(\alpha), f^{[k+1]}(\alpha)) \text{ and } (f_1(f^{[k+1]}(\alpha)), f_2(f^{[k+1]}(\alpha)))$$

are the pairs of divisors of t in $\boldsymbol{R}$.

Proof. a) follows from $f_*(t) = f(t)$ for every $t \in R$.
b) Since t is an f-primitive term in $\boldsymbol{R}$, $t \neq f(\alpha)$ for every $\alpha \in R$ and $t = u * v = uv$ for some $u, v \in R$. So, (u, v) is the unique pair of divisors of t in $\boldsymbol{R}$.
c) If $t = f(\alpha)$, where α is f-primitive in $\boldsymbol{R}$, then $t = (f_1 f_2)(\alpha) = f_1(\alpha) f_2(\alpha) = f_1(\alpha) * f_2(\alpha)$. Hence, $(f_1(\alpha), f_2(\alpha))$ is the unique pair of divisors of t in $\boldsymbol{R}$.
d) Let $t = f^{[k+2]}(\alpha)$, where $k \geq 0$ and α is an f-primitive term in $\boldsymbol{R}$. Then:

$$t = f^{[k+2]}(\alpha) = f(f(f^{[k]}(\alpha))) = f^{[k]}(\alpha) * f(f^{[k]}(\alpha)) = f^{[k]}(\alpha) * f^{[k+1]}(\alpha),$$

and so $(f^{[k]}(\alpha), f^{[k+1]}(\alpha))$ is a pair of divisors of t in $\boldsymbol{R}$. Also,

$$t = f^{[k+2]}(\alpha) = f(f^{[k+1]}(\alpha)) = (f_1 f_2)(f^{[k+1]}(\alpha)) =$$
$$= (f_1(f^{[k+1]}(\alpha)))(f_2(f^{[k+1]}(\alpha))) = (f_1(f^{[k+1]}(\alpha))) * (f_2(f^{[k+1]}(\alpha))),$$

and so $(f_1(f^{[k+1]}(\alpha)), f_2(f^{[k+1]}(\alpha)))$ is another pair of divisors of t in $\boldsymbol{R}$.

Definition 4.8 A groupoid $\boldsymbol{H} = (H, \cdot)$ is said to be $\mathcal{V}_f$-*injective* if it satisfies the following conditions:

(0) $\boldsymbol{H} \in \mathcal{V}_f$.

(1) For every $a \in H$ there is a unique f-primitive element $b \in H$ and a unique $k \geq 0$ such that $a = f^{[k]}(b)$.

In that case we say that b is an f-*base* of a, denoted by $\underline{a} = b$, $f^{[k]}$ is a k-*th* f-*power* of a, and k is an f-*exponent* of a.

(2) If a is a non-prime f-primitive element in $\boldsymbol{H}$, then there is a unique pair $(c, d) \in H \times H$ such that $a = cd$. We denote $(c, d)|a$.

(3) If $a = f(b)$, where $f = f_1 f_2$ and b is an f-primitive element in $\boldsymbol{H}$, then $(f_1(b), f_2(b))$ is the unique pair of divisors of a in $\boldsymbol{H}$.

(4) If $a = f^{[k+2]}(b)$, $k \geq 0$ and b is an f-primitive element in $\boldsymbol{H}$, then a has two pairs of divisors in $\boldsymbol{H}$: $(f^{[k]}(\alpha), f^{[k+1]}(\alpha))$ and $(f_1(f^{[k+1]}(\alpha)), f_2(f^{[k+1]}(\alpha)))$.

Since $R \in \mathcal{V}_f$, by Prop.4.4 and Prop.4.7, it follows that the groupoid $\boldsymbol{R} = (R, *)$ satisfies the conditions $(0)-(4)$ from the Definition 4.8. Since every $\mathcal{V}_f$-free groupoid over X is isomorphic with $\boldsymbol{R}$ it follows that

Proposition 4.9 *The class of $\mathcal{V}_f$-free groupoids is a subclass of the class of $\mathcal{V}_f$-injective groupoids.*

Lemma 4.10 *Let $\boldsymbol{H} = (H, \cdot)$ be a $\mathcal{V}_f$-injective groupoid such that the set P of primes in $\boldsymbol{H}$ is nonempty and generates $\boldsymbol{H}$. If $C_0 = P$, $C_1 = PP$, $C_{k+1} = \{a \in H \setminus P : (c, d)|\, a \Rightarrow \{c, d\} \subset C_0 \cup \ldots \cup C_k \,\wedge\, \{c, d\} \cap C_k \neq \emptyset\}$, then $H = \bigcup\{C_k : k \geq 0\}$, where $C_k \neq \emptyset$ for any $k \geq 0$, and $C_i \cap C_j = \emptyset$ for $i \neq j$.*

Proof. If $k = 1$ in C_{k+1}, then $a \in C_2$ if and only if $a \in H \setminus P$. This means that $a = cd$ and $c, d \in C_0 \cup C_1$, where at least one of c, d belongs to C_1. Three instances are possible: $cd \in C_0 C_1$, $cd \in C_1 C_0$, $cd \in C_1 C_1$. Thus, $a \in C_2$ if and only if $a \in C_0 C_1 \cup C_1 C_0 \cup C_1 C_1$, i.e. $C_2 = C_0 C_1 \cup C_1 C_0 \cup C_1 C_1$. By induction on k one obtains that the equality $C_{k+1} = C_0 C_k \cup C_k C_0 \cup C_1 C_k \cup C_k C_1 \cup \ldots \cup C_{k-1} C_k \cup C_k C_{k-1} \cup C_k C_k$ is true. Since the set P of primes in $\boldsymbol{H}$ is nonempty and generates $\boldsymbol{H}$, it follows that $H = \bigcup\{P_k : k \geq 0\}$, where $P_0 = P$, $P_{k+1} = P_k \cup P_k P_k$. By induction on k one obtains that $P_k = C_0 \cup C_1 \cup \ldots \cup C_k$, and as a consequence one obtains that $H = \bigcup\{C_k : k \geq 0\}$, where the union is disjoint.

We will use this lemma in the proof of the following theorem.

Theorem 4.11 (Bruck Theorem for $\mathcal{V}_f$) *A groupoid $\boldsymbol{H} = (H, \cdot)$ is $\mathcal{V}_f$-free if and only if it satisfies the following conditions:*

(i) $\boldsymbol{H}$ if $\mathcal{V}_f$-injective.

(ii) The set of primes in $\boldsymbol{H}$ is nonempty and generates $\boldsymbol{H}$.

Then the set P of primes in $\boldsymbol{H}$ is the unique $\mathcal{V}_f$-free generating set of $\boldsymbol{H}$.

Proof. If $\boldsymbol{H}$ is a $\mathcal{V}_f$-free groupoid over X, then by Prop.4.9 it follows that the groupoid $\boldsymbol{H}$ is $\mathcal{V}_f$-injective. Since every $\mathcal{V}_f$-free groupoid over X is isomorphic with $\boldsymbol{R}$ and by the proof of Theorem 3.4, it follows that X is the set of primes in $\boldsymbol{H}$ that is nonempty and generates $\boldsymbol{H}$.

Conversely, let (i) and (ii) hold. It suffices to show that $\boldsymbol{H}$ has the universal mapping property for $\mathcal{V}_f$ over the set P of primes in $\boldsymbol{H}$. For that reason, define an infinite sequence of subsets $C_0, C_1, \dots$ of $\boldsymbol{H}$ by: $C_0 = P$, $C_1 = PP$, $\dots$, $C_{k+1} = \{a \in H \setminus P : (c,d)|\, a \Rightarrow \{c,d\} \subset C_0 \cup \ldots \cup C_k \ \wedge \ \{c,d\} \cap C_k \neq \emptyset\}$, as in Lemma 4.10 Then, $C_i \neq \emptyset$ and $H = \bigcup\{C_k : \ k \geq 0\}$, where $C_i \cap C_j = \emptyset$, for any i, j, $i \neq j$.

Let $\boldsymbol{G} \in \mathcal{V}_f$ and $\lambda : P \to G$ is a mapping. For every nonnegative integer k define a mapping $\varphi_k : C_k \to G$ by $\varphi_0 = \lambda$ and let φ_i be defined for every $i \leq k$. Let $a \in C_{k+1}$, $(c,d)|a$ and $c \in C_r$, $d \in C_s$. Then $r, s \leq k$. Putting $\varphi_{k+1}(a) = \varphi_r(c)\varphi_s(d)$ we obtain that $\varphi = \bigcup\{\varphi_i : \ \ i \geq 0\}$ is a well defined mapping from H into G. By induction on k one can show that $\varphi(f^{[k]}(a)) = f^{[k]}(\varphi(a))$, for every $a \in H$.

Let a be a non-prime f-primitive element in $\boldsymbol{H}$ or $a = f(b)$, where b is an f-primitive element in $\boldsymbol{H}$ and $f = f_1 f_2$. Then there is a unique pair $(c,d) \in H \times H$ such that $a = cd$. In these cases $\varphi(a) = \varphi(cd) = \varphi(c)\varphi(d)$.

If $a = f^{[k+2]}(b)$, where b is an f-primitive element in $\boldsymbol{H}$, then consider two cases: 1) $c = f^{[k]}(b)$, $d = f^{[k+1]}(b)$ and 2) $c = f_1(f^{[k+1]}(b))$, $d = f_2(f^{[k+1]}(b))$.

In the first case we obtain:

$$\varphi(cd) = \varphi(f^{[k+2]}(b)) = f^{[k+2]}(\varphi(b)) = [\boldsymbol{G} \in \mathcal{V}_f] = \\ = f^{[k]}(\varphi(b)) f^{[k+1]}(\varphi(b)) = \varphi(f^{[k]}(b))\varphi(f^{[k+1]}(b)) = \varphi(c)\varphi(d).$$

In the second case:

$$\varphi(cd) = \varphi(f^{[k+2]}(b)) = f^{[k+2]}(\varphi(b)) = f(f^{[k+1]}(\varphi(b))) = \\ = (f_1 f_2)(f^{[k+1]}(\varphi(b))) = f_1(f^{[k+1]}(\varphi(b)))\, f_2(f^{[k+1]}(\varphi(b))) = \\ = \varphi(f_1(f^{[k+1]}(b)))\varphi(f_2(f^{[k+1]}(b))) = \varphi(c)\varphi(d).$$

Hence, φ is a homomorphism from $\boldsymbol{H}$ into $\boldsymbol{G}$. Therefore, $\boldsymbol{H}$ is a $\mathcal{V}_f$-free groupoid over X.

There are $\mathcal{V}_f$-injective groupoids that are not $\mathcal{V}_f$-free, as the following example shows.

Example 4.12 Let X be a countable set and let $\boldsymbol{R} = (R, *)$ be the canonical groupoid in $\mathcal{V}_f$ over X. Define two sets $H \subseteq R$, $D \subseteq H \times H$ by:

$H = \{w \in R : \ |cn(w)| = 1\}$ and $D = \{(u,v) \in H \times H : cn(u) \neq cn(v)\}$.

Note that $H = \bigcup_{x \in X} \langle\, x \,\rangle_*$.

The sets D and X have the same cardinality and therefore there is an injection $\varphi : D \to X$. Define an operation $\otimes$ in H by:

$$u, v \in H \ \Rightarrow \ u \otimes v = \begin{cases} u * v, & \text{if } cn(u) = cn(v), \\ \varphi(u,v), & \text{if } cn(u) \neq cn(v). \end{cases}$$

The operation $\otimes$ is well defined, i.e. $\boldsymbol{H} = (H, \otimes)$ is a groupoid. Directly from the definition of $\otimes$ it follows that $\boldsymbol{H} \in \mathcal{V}_f$. By Cor.4.5 it follows that (1) from Definition 4.8 holds. The groupoid $\boldsymbol{H} = (H, \otimes)$ satisfies the conditions (2) – (4) from Definition 4.8, by Prop.4.7 b), c), d). Therefore, the groupoid $\boldsymbol{H}$ is $\mathcal{V}_f$-injective. If $\varphi : D \to X$ is a bijection, then $X \setminus im\varphi$ is an empty set, i.e. the set of primes in $(H, \otimes)$ is empty. By Bruck Theorem for $\mathcal{V}_f$ we obtain that the groupoid $(H, \otimes)$ is not $\mathcal{V}_f$-free.

Thus, we have proved the following

Proposition 4.13 *The class of $\mathcal{V}_f$-free groupoids is a proper subclass of the class of $\mathcal{V}_f$-injective groupoids.*

Hence, the characterization of $\mathcal{V}_f$-free objects by means of $\mathcal{V}_f$-injective groupoids is completed.

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