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Prime Ideal Spaces of A*-Algebras

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Abstract

In his paper **[5]**, M. H. Stone introduced and studied the prime ideal spaces of Boolean Algebras. In this paper we introduce the prime ideal spaces of A^*-algebras, prove the space is a totally disconnected compact Hausdorff space and every Boolean element of A corresponds to a clopen subset of the space and prove that every A^*- algebra is isomorphic to the A^*- algebra generated by the class of all clopen subsets of a totally disconnected compact Hausdorff space.

Keywords: Boolean algebra, A*-algebra, A*-ideal, prime ideal

1. Preliminaries

1.1 Definition: - A Boolean algebra is an algebra $(B,\vee, \wedge, (-)', 0, 1)$ with two binary operations, one unary operation (called complementation), and two nullary operations which satisfies:

(1) $(B,\vee, \wedge)$ is a distributive lattice.

(2) $x \wedge 0 = 0, \quad x \vee 1 = 1$

(3) $x \wedge x' = 0, x \vee x' = 1$

1.2 Definition: An algebra $(A, \wedge, *, (-)^{\sim}, (-)_{\pi}, 1)$ is an A* - algebra if it satisfies **:**
For a, b, $c \in A$

(i) $a_{\pi} \vee (a_{\pi})^{\sim} = 1$, $(a_{\pi})_{\pi} = a_{\pi}$, where $a \vee b = (a^{\sim} \wedge b^{\sim})^{\sim}$.

(ii) $a_{\pi} \vee b_{\pi} = b_{\pi} \vee a_{\pi}$

(iii) $(a_{\pi} \vee b_{\pi}) \vee c_{\pi} = a_{\pi} \vee (b_{\pi} \vee c_{\pi})$

(iv) $(a_{\pi} \wedge b_{\pi}) \vee (a_{\pi} \wedge (b_{\pi})^{\sim}) = a_{\pi}$

(v) $(a \wedge b)_{\pi} = a_{\pi} \wedge b_{\pi}$, $(a \wedge b)^{\#} = a^{\#} \vee b^{\#}$,where $a^{\#} = (a_{\pi} \vee a^{\sim}{}_{\pi})^{\sim}$

(vi) $a^{\sim}{}_{\pi} = (a_{\pi} \vee a^{\#})^{\sim}$, $a^{\sim\#} = a^{\#}$

(vii) $(a*b)_{\pi} = a_{\pi}$, $(a*b)^{\#} = (a_{\pi})^{\sim} \wedge (b^{\sim}{}_{\pi})^{\sim}$

(viii) $a = b$ if and only if $a_{\pi} = b_{\pi}$, $a^{\#} = b^{\#}$. We write 0 for $1^{\sim}$, 2 for $0*1$.

1.3 Example: 3 =$\{0,1,2\}$ with the operations defined below is an A* -algebra

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$*$	0	1	2
0	0	2	2
1	1	1	1
2	0	2	2

x	0	1	2
$x^{\sim}$	1	0	2
x_{π}	0	1	0
$x^{\#}$	0	0	1

1.4 Huntington's Theorem: Let B have one binary operation $\vee$and one unary operations $(-)'$ and define

(i) $a \wedge b = (a' \vee b')'$ for all $a, b \in B$.

Suppose for all a, b, $c \in B$,

(ii) $a \vee b = b \vee a$.

(iii) $a \vee (b \vee c) = (a \vee b) \vee c$.

(iv) $(a \wedge b) \vee (a \wedge b') = a$

Then B is a Boolean algebra.

1.5. Note: From 1.2 (i) through (iv) and by 1.4, $B(A) = \{ a_\pi / a \in A\}$ is a Boolean algebra with $\wedge,\vee,(-)^\sim$, 0 and $a \in B(A) \Rightarrow a_\pi = a$. Since $1, 0, (a_\pi)^\sim \in B(A)$, we have $1_\pi = 1$, $0_\pi = 0$, $(a_\pi)^\sim{}_\pi = (a_\pi)^\sim$ and $a_\pi \wedge a^\# = 0$, $a * 0 = a_\pi$.

1.6. Theorem: Let $(B, \wedge, (-)', 0)$ be a Boolean algebra. Then $A(B) = \{(a, b)/a, b \in B, a \wedge b = 0\}$ becomes an A* -algebra, where the A*-algebraic operations $\wedge, \vee, *, (-)^\sim, (-)_\pi$ are defined as follows:

For $a = (x_\pi , x^\#)$, $b = (y_\pi, y^\#) \in A(B)$

(i) $a \wedge b = (x_\pi y_\pi , x_\pi y^\# + x^\# y_\pi + x^\# y^\#)$
(ii) $a \vee b = (x_\pi y_\pi + x_\pi y^\# + x^\# y_\pi, x^\# y^\#)$
(iii) $a\sim \ = (x^{\#,}\ x_\pi)$
(iv) $a_\pi = (x_\pi , (x_\pi)')$
(v) $a*b = (x_\pi, (x_\pi)'y\#)$
(vi) $1 = (1,0)$, $0 = (0,1)$, $2 = (0,0)$
(where juxta position ,+,$(-)'$ respectively $\wedge,\vee,(-)'$ in Boolean algebra B).

1.7. Theorem [1] : If A is an A* -algebra and B is a Boolean algebra then
(i) $B (A (B)) \cong B$
(ii) $A (B(A)) \cong A$

1.8. Theorem [1]: Let A_1, A_2 be A*- algebras and B_1, B_2 be Boolean algebras, then
(i) $A_1 \cong A_2$ iff $B (A_1) \cong B (A_2)$.
(ii) $B_1 \cong B_2$ iff $A (B_1) \cong A (B_2)$.

1.9. Definition : Let $(A_1, \wedge,\vee,(-)^\sim, (-)_\pi , *,1)$ and $(A_2,\wedge,\vee, (-)^\sim,(-)_\pi , *,1)$ be A*-algebras. A Mapping $f: A_1 \rightarrow A_2$ is called an A* - homomorphism if for all $a, b \in A_1$
(i) $f(a\wedge b) = f(a)\wedge f(b)$
(ii) $f(a\vee b) = f(a)\vee f(b)$
(iii) $f(a * b) = f(a) * f(b)$
(iv) $f(a_\pi) = (f(a))_\pi$
(v) $f(a^\sim) = (f(a))^\sim$
(vi) $f(1) = 1$
(vii) $f(0) = 0$.

If in addition f is bijective, then f is called an A*- isomorphism and is denoted by $A_1 \cong A_2$.

1.10. Definition: A nonempty subset I of an A^*-algebra A is said to be an A^*-ideal if
(i) $a, b \in I \Rightarrow a \vee b, a * b \in I$.
(ii) $a \in I \Rightarrow a_\pi, a^\# \in I$
(iii) $a \in I, b \in A \Rightarrow a_\pi b_\pi, a^\# b^\# \in I$
(Here $xy = x \wedge y$ for all $x, y \in B(A)$).

1.11. Theorem [3]: Suppose θ is a congruence relation on A. Then $\theta(0)$ is an A*-ideal.

1.12. Theorem [3]: Suppose A is an A*- algebra and B = B(A). Then there is a one-to-one correspondence between the set of all ideals of A and the set of all ideals of B.

1.13. Note: (i) $a\Delta b = \{ a_\pi b_\pi, a_\pi b^\#, a^\# b_\pi, a^\# b^\# \}$
(ii) if A is an A*-algebra and a, b, $a_i \in A$ then $a^0 = a^{\sim}_\pi$, $a^1 = a_\pi$, $a^2 = a^\#$.
(iii) $a^1 a^0 = a^2 a^1 = a^0 a^2 = 0$
(iv) $a^{01} = a^0$, $a^{11} = a^1$, $a^{21} = a^2$, $a^{00} = a^{0\sim}$, $a^{10} = a^{1\sim}$, $a^{20} = a^{2\sim}$, $a^{02} = 0$,
$a^{12} = 0$, $a^{22} = 0$.

1.14. Definition [3]: An ideal I of an A^*-algebra A is said to be prime ideal if for any $a, b \in A$, $a\Delta b \subseteq I \Rightarrow a \in I$ or $b \in I$.

1.15. Theorem [3]: An ideal I of an A*-algebra is prime ideal iff $I \cap B$ is Boolean prime ideal, where B=B(A).

1.16. Theorem [3]: Suppose A is an A^*-algebra and f: $A \rightarrow$ **3**= {0, 1, 2} is an A*-homomorphism. Then ker f=$\{a \in A / f(a) = 0\}$ is a prime ideal.

1.17. Theorem [3]: In an A*-algebra A every prime ideal is a maximal ideal.

1.18. Theorem [3]: An ideal I of an A*-algebra A is maximal ideal iff $I \cap B$ is Boolean maximal ideal, where B= B(A).

1.19. Theorem :(i) P is prime ideal and $e, f \in B(A)$, $ef \in P \Rightarrow e \in P$ or $f \in P$
(ii) P is prime ideal and for $a \in A$, $a_\pi \in P$ or $a^\# \in P$.

Proof: (i) $e\Delta f = \{e^1 f^1, e^2 f^1, e^1 f^2, e^2 f^2\}$ (by 1.13)
$= \{ef, 0,0,0\}$
$= \{ef, 0\} \subseteq P$
$\Rightarrow e \in P$ or $f \in P$
Therefore $ef \in P \Rightarrow e \in P$ or $f \in P$.
(ii) Since $a_\pi a^\# = 0 \in P \Rightarrow a_\pi \in P$ or $a^\# \in P$.

1.20. Note: Suppose A is an A*- algebra and $C \subseteq A$. The ideal generated by C is denoted by Gen(C) is defined as

$$\mathrm{Gen}(C) = \{ \bigvee_{1 \le p \le q} [(a_p^{\alpha_p} \wedge b_p^{\gamma_p}) * (a_p^{\beta_p} \wedge b_p^{\theta_p})] / a_p \in C,\ b_p \in A,\ \alpha_p, \beta_p, \gamma_p, \theta_p \in \mathbf{3}\}$$

1.21. Note: For every finite sequence $a_1, a_2, .., a_n$ of C, $a_1 \vee a_2 \vee \ldots\ldots \vee a_n \neq 1$ Gen(C) is a proper ideal.

1.22. Theorem [1]: Every proper ideal I of A is contained in a maximal ideal.

2. Prime ideal Spaces

2.1 Theorem: In an A*-algebra every maximal ideal is a prime ideal.

Proof: Suppose M is a maximal ideal.

$\Rightarrow$ M_π is a Boolean maximal ideal

$\Rightarrow$ M_π is a Boolean prime ideal

Let a, $b \in A$ and $a\Delta b \subseteq M$.

Suppose $a \notin M$.

$\Rightarrow a_\pi \notin M$ or $a^\# \notin M$

Suppose $a_\pi \notin M$

Since $a\Delta b \subseteq M$,i.e., $\{ a_\pi b_\pi,\ a_\pi b^\#,\ a^\# b_\pi,\ a^\# b^\# \} \subseteq M$

$\Rightarrow a_\pi b_\pi,\ a_\pi b^\# \in M_\pi$

$\Rightarrow$ $b_\pi, b^\# \in M_\pi$ ($\because M_\pi$ is prime)

$\Rightarrow$ $b_\pi * b^\# \in M$

$\Rightarrow$ $b \in M$

Therefore M is a prime ideal.

2.2. Definition: Suppose $(A, \wedge, \vee, (-)^{\sim}, (-)_\pi, *, 0,1,2)$ is an A^*-algebra . Let X be the set of all prime ideals of A(X is also denoted by Spec A). Let $a \in A$, $C \subseteq A$.
$X_a = \{P \in X / a \notin P\}$ and $X_C = \{P \in X / C \not\subseteq P\}$.

2.3 Note: (i) $X_1 = \{P \in X / 1 \notin P\} = X$

(ii) $X_0 = \{P \in X / 0 \notin P\} = \phi$

(iii) For a, $b \in A$, $a\Delta b = \{a^1 b^1, a^1 b^2, a^2 b^1, a^2 b^2\}$.

2.4 Lemma: Let $a,b \in A$, $C \subseteq A$. Then

(i) $X_{a\Delta b} = X_a \cap X_b$

(ii) $X_C = \bigcup_{a \in C} X_a$

Proof: (i) $X_{a\Delta b} = \{P \in X / a\Delta b \not\subseteq P\}$

$= \{P \in X / a \notin P \text{ and } b \notin P\}$

$=\{P\in X/ a\notin P\} \cap \{P\in X/b\notin P\}$
$= X_a \cap X_b$.

(ii) $P\in X_C \Leftrightarrow C\not\subseteq P$
$\Leftrightarrow a\notin P$ for some $a\in C$
$\Leftrightarrow P\in X_a$ for some $a\in C$
$\Leftrightarrow P\in \bigcup_{a\in C} X_a$

Therefore $X_C = \bigcup_{a\in C} X_a$

2.5 Definition: A, X as in 2.2. From 2.3(i) and (ii), $\{X_a / a\in A\}$ forms an open base for which X is a topological space. This topological space X is called prime ideal space of A.

2.6 Note: $C\subseteq A$, $D\subseteq A$

$$C\Delta D = \bigcup_{\substack{a\in C\\ b\in D}} a\Delta b.$$

2.7 Lemma: $X_C \cap X_D = X_{C\Delta D}$

Proof: $P\in X_C \cap X_D \Leftrightarrow P\in X_C$ and $P\in X_D$
$\Leftrightarrow C\not\subseteq P$ and $D\not\subseteq P$
$\Leftrightarrow \exists\ a\in C, b\in D \ni a\notin P$ and $b\notin P$
$\Leftrightarrow a\Delta b\not\subseteq P$
$\Leftrightarrow C\Delta D\not\subseteq P$
$\Leftrightarrow P\in X_{C\Delta D}$

Therefore $X_C \cap X_D = X_{C\Delta D}$.

2.8 Note: $X_{A\cup B} = X_A \cup X_B$.

2.9 Lemma: For every $e\in B(A)$, X_e is a clopen set.

Proof: $X_e = \{P\in X/e\notin P\}$
$X_{e^\sim} = \{P\in X/e^\sim\notin P\}$
$= \{P\in X/e\in P\}^c$
$= (X_e)^c$

That is $(X_e)^c$ is open set
ThenX_e is Closed Set.
Therefore X_e is clopen set.

2.10 Theorem: Suppose A is an A^*-algebra. Then X, the Prime ideal space of A is totally disconnected.

Proof: Suppose $P_1, P_2 \in X$ & $P_1 \neq P_2$

Then P_1, P_2 are two distinct Prime ideals of A

Let $a \in P_1 - P_2$.

Then $a \in P_1$ and $a \notin P_2$

Since $a \in P_1 \Rightarrow a_\pi \in P_1$, $a^\# \in P_1$

Since $a \in A$ and P_2 is Prime ideal so $a_\pi \in P_2$ or $a^\# \in P_2$.

Suppose $a_\pi \in P_2 \Rightarrow a^\# \notin P_2$

Therefore $a^\# \in P_1$, $a^\# \notin P_2$

$\Rightarrow a^{\#\sim} \notin P_1$, $a^\# \notin P_2$

$\Rightarrow P_1 \in X_a^{\#\sim}$, $P_2 \in X_a^{\#}$

$X_{a\#\sim} \cap X_{a\#} = X_{a\# \Delta a\#\sim}$

$= X_0 = \phi$

And $X_a^{\#\sim} \cup X_a^{\#} = X_{a^{\#\sim} \vee a^{\#}}$

$= X_1 = X$.

Therefore for $P_1, P_2 \in X$ with $P_1 \neq P_2$, $\exists$ two disjoint open sets $X_a^{\#\sim}$, $X_a^{\#}$ such that $X_a^{\#\sim} \cup X_a^{\#} = X$.

Suppose $a_\pi \notin P_2$

Then $a_\pi^{\sim} \notin P_1$ and $a_\pi \notin P_2$

$\Rightarrow P_1 \in X_{a_\pi}^{\sim}$, $P_2 \in X_{a_\pi}$ and $X_{a_\pi}^{\sim} \cap X_{a_\pi} = \phi$, $X_{a_\pi}^{\sim} \cup X_{a_\pi} = X$.

Therefore X is totally disconnected.

Since X is totally disconnected, X is Hausdorff Space.

2.11 Theorem: X is Compact space.

Proof: Suppose $\{X_a / a \in C\}$ is a basic open Cover for X where $C \subseteq A$.

Then $X = \bigcup_{a \in C} X_a$.

Suppose there is no finite sequence $a_1, a_2, \ldots, a_n \in C$ such that $a_1 \vee a_2 \vee \ldots \vee a_n \neq 1$. Then the ideal generated by C i.e.,$\langle C \rangle$ is a proper ideal of A.

Then $\exists$ a maximal ideal M of A such that $\langle C \rangle \subseteq M$.

Since M is maximal ideal of R, M is a prime ideal of A.

Since $C \subseteq M \Rightarrow a \in M\ \forall a \in C$

$\Rightarrow M \notin X_a \forall a \in C$

$\Rightarrow M \notin \bigcup_{a \in C} X_a$

$\Rightarrow M \notin X$.

It is a contradiction.

Therefore there exists $a_1, a_2, \ldots, a_n \in C$ such that $a_1 \vee a_2 \vee \ldots \vee a_n = 1$.

Then $X = X_1 = X_{a_1 \vee a_2 \vee \ldots\ldots\ldots \vee a_n}$

$= X_{a_1} \cup X_{a_2} \cup .... \cup X_{a_n}$

That is the basic open cover $\{X_a/a\in C\}$ has a finite sub-cover for X.
Therefore X is Compact.

2.12 Theorem: Suppose A is an A^*-algebra and $a\in A$, then

$X_a = X_{a_\pi} \cup X_{a^\#} = X_{a_\pi \vee a^\#} = X_e$, where $e = a_\pi \vee a^\# \in B(A)$

Proof: $X_a = \{P\in X/a\notin P\}$

$= \{P\in X/ a_\pi\notin P \text{ or } a^\# \notin P\}$

$= \{P\in X/ a_\pi\notin P\}\cup\{P\in X/ a^\# \notin P\}$

$= X_{a_\pi} \cup X_{a^\#}$

$= X_{a_\pi \vee a^\#}$

Therefore $X_a = X_{a_\pi \vee a^\#} = X_e$, $e = a_\pi \vee a^\# \in B(A)$

2.13 Theorem: If Y is a clopen subsets of X then $\exists\, e\in B(A)$ such that $X_e = Y$.

Proof: Let Y is a Clopen subset of X

Since Y is closed and X is Compact $\exists\, a_1,a_2,\ldots\ldots\ldots,a_n\in R \ni$

$Y = X_{a_1} \cup X_{a_2} \cup .... \cup X_{a_n}$

$= X_{a_1 \vee a_2 \vee .... \vee a_n}$

$= X_a$ where $a = a_1\vee a_2\vee\ldots\ldots\vee a_n$

Therefore $Y = X_a$

Since $X_a = X_{a_\pi \vee a^\#} = X_e$ where $e = a_\pi \vee a^\# \in B(A)$

Therefore $Y = X_e$ for some $e\in B(A)$.

2.14 Theorem: B(A) and $\{X_e / e\in B(A)\}$ are Boolean isomorphic.

Proof: Proof is obvious.

2.15 Theorem: Every A^*-algebra A and an A^*-algebra generated by clopen subsets of totally-disconnected Compact Hausdorffspace are isomorphic.

Proof: Suppose $(A, \wedge,\vee,(-)^{\sim},(-)_\pi ,*, 0,1,2)$ is an A^*-algebra . Let X be the set of all prime ideals of A.

Then by Theorems 2.10& 2.11, X is a totally disconnected Compact Hausdorff -Space.

Let $B=B(A)$, $X_B =\{X_e/e\in B(A)\}$

By the above, $B \cong X_B$.

Then $A(X_B)$ is the A^*-algebra generated by X_B.

Since A (B) ≅ A, A(B) ≅ A(X_B), then A≅ A(X_B).
Hence the theorem.

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