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On the Vanishing and Finiteness Properties of Generalized Local Cohomology Modules

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Abstract

Let R be a commutative noetherian ring, $\mathfrak{a}$ an ideal of R and M, N finite R–modules. We prove that the following statements are equivalent.

(i) $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is finite for all $i < n$.

(ii) $\mathrm{Coass}_R(\mathrm{H}^i_{\mathfrak{a}}(M,N)) \subset \mathrm{V}(\mathfrak{a})$ for all $i < n$.

(iii) $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is coatomic for all $i < n$.

If $\mathrm{pd}\, M$ is finite and r be a non-negative integer such that $r > \mathrm{pd}\, M$ and $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is finite (resp. minimax) for all $i \geq r$, then $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is zero (resp. artinian) for all $i \geq r$.

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1 Introduction

Throughout R is a commutative noetherian ring. Generalized local cohomology was given in the local case by J. Herzog [5] and in the more general case by M.H Bijan-Zadeh [2]. Let $\mathfrak{a}$ denote an ideal of a ring R. The generalized local cohomology defined by

$$\mathrm{H}^i_{\mathfrak{a}}(M,N) \cong \varinjlim_n \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N).$$

This concept was studied in the articles [8], [5] and [9]. Note that this is in fact a generalization of the usual local cohomology, because if $M = R$, then $\mathrm{H}^i_{\mathfrak{a}}(R,N) = \mathrm{H}^i_{\mathfrak{a}}(N)$. Important problems concerning local cohomology are vanishing, finiteness and artinianness results (see [6]).

In Section 2 we show in 2.1 that if M is finite and all generalized local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ are coatomic for all $i < n$, then they are finite for all $i < n$. In fact this is another condition equivalent to Falting's Local-global Principle for the finiteness of generalized local cohomology modules (see [1, Theorem 2.9]). In Theorem 2.2 we generalize Yoshida's theorem ([10, Theorem 3.1]).

In Section 3, We prove in 3.2, that when M is a finite R–module of finite projective dimension such that the generalized local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ are minimax modules for all $i \geq r$, (where $r > \operatorname{pd} M$) then they must be artinian.

For unexplained terminology we refer to [3] and [4].

2 Finiteness and vanishing

An R–module M is called *coatomic* when each proper submodule N of M is contained in a maximal submodule N' of M (i.e. such that $M/N' \cong R/\mathfrak{m}$ for some $\mathfrak{m} \in \operatorname{Max} R$). This property can also be expressed by $\operatorname{Coass}_R(M) \subset \operatorname{Max} R$ or equivalently that any artinian homomorphic image of M must have finite length. In particular all finite modules are coatomic. Coatomic modules have been studied by Zöschinger [12].

Theorem 2.1 *Let R be a noetherian ring, $\mathfrak{a}$ an ideal of R and M, N finite R–modules. The following statements are equivalent:*

(i) $H^i_{\mathfrak{a}}(M,N)$ is coatomic for all $i < n$.

(ii) $\operatorname{Coass}_R(\mathrm{H}^i_{\mathfrak{a}}(M,N)) \subset \mathrm{V}(\mathfrak{a})$ for all $i < n$.

(iii) $H^i_{\mathfrak{a}}(M,N)$ is finite for all $i < n$.

proof: *By [1, Theorem 2.9] and[12, 1.1, Folgerung] we may assume that $(R,\mathfrak{m})$ is a local ring.*

(i) $\Rightarrow$ (ii) It is trivial by the definition of coatomic modules.

(ii) $\Rightarrow$ (iii) By [15, Satz 1.2] there is $t \geq 1$ such that $\mathfrak{a}^t\,\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is finite for all $i < n$. Therefore there is $s \geq t$ such that $\mathfrak{a}^s\,\mathrm{H}^i_{\mathfrak{a}}(M,N) = 0$ for all $i < n$, and apply [1, Theorem 2.9].

(iii) $\Rightarrow$ (i) Any finite R–module is coatomic.

The following results are generalizations of [10, Proposition 3.1].

Theorem 2.2 *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ be an ideal of R and M be a finite module of finite projective dimension. Let N be a finite module and $r > \operatorname{pd} M$. If $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is finite for all $i \geq r$, then $\mathrm{H}^i_{\mathfrak{a}}(M,N) = 0$ for all $i \geq r$.*

proof: *We prove by induction on $d = \dim N$. If $d = 0$, By [9, Theorem 3.7], it follows that $\mathrm{H}^i_{\mathfrak{a}}(M,N) = 0$ for all $i > \operatorname{pd} M + \dim(M \otimes_R N)$ and so the claim clearly holds for $n = 0$. Now suppose $d > 0$ and $\mathrm{H}^i_{\mathfrak{a}}(M,N) = 0$ for all $i > r$. It is enough to show $\mathrm{H}^r_{\mathfrak{a}}(M,N) = 0$. First suppose* $\operatorname{depth}_R N > 0$. *Take $x \in \mathfrak{m}$ which is N–regular. Then* $\dim N/xN = d - 1$. *The exact sequence*

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$$

induces the exact sequence

$$\mathrm{H}^r_{\mathfrak{a}}(M,N) \xrightarrow{x} \mathrm{H}^r_{\mathfrak{a}}(M,N) \longrightarrow \mathrm{H}^r_{\mathfrak{a}}(M,N/xN) \longrightarrow \mathrm{H}^{r+1}_{\mathfrak{a}}(M,N) = 0$$

It yields that $\mathrm{H}^i_{\mathfrak{a}}(M,N/xN) = 0$ for all $i > r$. Hence by induction hypothesis we get $\mathrm{H}^r_{\mathfrak{a}}(M,N/xN) = 0$. Thus we have $\mathrm{H}^r_{\mathfrak{a}}(M,N) = 0$ by Nakayama's lemma. Next suppose $\operatorname{depth}_R N = 0$. *Put $L = \Gamma_{\mathfrak{m}}(N)$. Since L have finite length, so we have* $\dim L = 0$ *and therefore $\mathrm{H}^i_{\mathfrak{a}}(M,L) = 0$ for all $i > \operatorname{pd} M$. But from the exact sequence*

$$0 \longrightarrow L \longrightarrow N \longrightarrow N/L \longrightarrow 0$$

we get the exact sequence

$$\ldots \to \mathrm{H}^i_{\mathfrak{a}}(M,L) \to \mathrm{H}^i_{\mathfrak{a}}(M,N) \to \mathrm{H}^i_{\mathfrak{a}}(M,N/L) \to \mathrm{H}^{i+1}_{\mathfrak{a}}(M,L) \to \ldots$$

hence we have $\mathrm{H}^i_{\mathfrak{a}}(M,N) \cong \mathrm{H}^i_{\mathfrak{a}}(M,N/L)$ for all $i > \operatorname{pd} M$, and we get the required assertion from the first step.

Theorem 2.3 *Let $\mathfrak{a}$ be an ideal of R and M a finite R–module of finite projective dimension. Let N be a finite R–module and $r > \operatorname{pd} M$. The following statements are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(M,N) = 0$ *for all $i \geq r$.*

(ii) $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ *is finite for all $i \geq r$.*

(iii) $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ *is coatomic for all $i \geq r$.*

proof: *(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) Trivial. (iii) $\Rightarrow$ (i) By use of theorem 2.2 and [12, 1.1, Folgerung] we may assume that $(R, \mathfrak{m})$ is a local ring. Note that coatomic modules satisfy Nakayama's lemma. So the proof is the same as in theorem 2.2.*

In the following corollary $\operatorname{cd}_{\mathfrak{a}}(M,N)$ denote the supremum of i's such that $\mathrm{H}^i_{\mathfrak{a}}(M,N) \neq 0$.

Corollary 2.4 *Let $\mathfrak{a}$ an ideal of R, M a finite R–module of finite projective dimention and N a finite R–module. If $c := \operatorname{cd}_{\mathfrak{a}}(M,N) > \operatorname{pd} M$, then $\mathrm{H}^c_{\mathfrak{a}}(M,N)$ is not coatomic in particular is not finite.*

3 Artinianness

Recall that a module M is a *minimax* module if there is a finite (i.e. finitely generated) submodule N of M such that the quotient module M/N is artinian. Thus the class of minimax modules includes all finite and all artinian modules. Moreover, it is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of R–modules. Minimax modules have been studied by Zink in [11] and Zöschinger in [13, 14]. See also [7].

Lemma 3.1 *Let M and N be two R–module. If $f : R \longrightarrow S$ is a flat ring homomorphism, then*

$$\mathrm{H}^i_{\mathfrak{a}}(M,N) \otimes_R S \cong \mathrm{H}^i_{\mathfrak{a}} S(M \otimes_R S, N \otimes_R S).$$

proof: *It is easy and we lift it to the reader.*

Theorem 3.2 *Let $\mathfrak{a}$ an ideal of R and M a finite R–module of finite projective dimension. Let N be a finite R–module and $r > \operatorname{pd} M$. If $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is a minimax module for all $i \geq r$, then $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is an artinian module for all $i \geq r$.*

proof: *Let $\mathfrak{p}$ be a non-maximal prime ideal of R. Then by the definition of minimax module and lemma 3.1 $\mathrm{H}^i_{\mathfrak{a}}(M,N)_{\mathfrak{p}} \cong \mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ is a finite $R_{\mathfrak{p}}$–module for all $i \geq r$. By theorem 2.2, $\mathrm{H}^i_{\mathfrak{a}}(M,N)_{\mathfrak{p}} = 0$ for all $i \geq r$, thus $\mathrm{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M,N)) \subset \operatorname{Max} R$ for all $i \geq r$. By [7, Theorem 2.1], $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is artinian for all $i \geq r$.*

Let $q_{\mathfrak{a}}(M,N)$ denote the supremum of the i's such that $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is not artinian with the usual convention that the supremum of the empty set of integers is interpreted as $-\infty$.

Corollary 3.3 *Let $\mathfrak{a}$ an ideal of R, M a finite R–module of finite projective dimension and N a finite R–module. If $q := q_{\mathfrak{a}}(M,N) > \operatorname{pd} M$, then $\mathrm{H}^q_{\mathfrak{a}}(M,N)$ is not minimax in particular is not finite.*

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