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IC-Pseudo-Injective Modules

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Abstract: In this work, the notion of pseudo-injectivity relative to a class of submodules (namely, ic-pseudo-injectivity) has been introduced and studied, which is a proper generalization of pseudo-injectivity and continuity. This notion is closed under direct summands. Several properties and characterizations have been given. Continuous and quasi-continuous modules are characterized in terms of lifting monomorphisms of certain submodules into the module. Noetherian rings and semisimple artinian rings have been characterized in terms of ic-pseudo-injectivity.

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1 INTRODUCTION

Throughout, R represents an associative ring with identity and R-modules are unitary right R-modules.
Let M and N be two R-modules, N is called (pseudo)-M-injective, if for every submodule A of M,any R-homomorphism (R-monomorphism) from A to N can be extended to an R-homomorphism from M to N. An R-module is injective , if it is M-injective for all R-module M [6]. An R-module M is called quasi(pseudo)-injective, if it is (pseudo)-M-injective[6],[3].

A submodule N of an R-module M is essential, if N has non-trivial intersection with every non-zero submodule of M. A submodule K of an R-module

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M is closed, if K has no proper essential extension in M. An R-module M is called CS-module(extending), if M satisfies any one of the following conditions (1) : for every submodule N of M, there is a decomposition $M = A \oplus B$ such that N is essential in A, (2): every closed submodule of M is a direct summand. A CS-module M which satisfies condition (C_2): every submodule of M which is isomorphic to a direct summand of M is itself direct summand, is called continuous. It is well-known that condition (C_2) implies condition (C_3): If A and B be direct summands of an R-module M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M. A CS-module M which satisfies condition (C_3) is called quasi-continuous [6]. So we have the following implications:

$Continuous \Rightarrow quasi - continuous \Rightarrow CS - modules$

Generally neither of the converse implications is true. Among aims of this work establishing the converse of the above implications under ic-pseudo-injective modules. We introduce certain class of submodules (ic-submodules). By this class, we reformulate continuous modules, also we decompose projective modules over right hereditary pri rings, as a direct sum of noetherian uniform modules. Also, we characterize continuous modules as modules whose closed submodules are ic-pseudo-injective. Several results on pseudo-injective modules have been extended to ic-pseudo-injective modules. Noetherian rings are characterize as every f-injective module is ic-pseudo-injective Also we invst ic-pseudo-injectivity to characterize semisimple artinian rings.

2 RELATIVE IC-PSEUDO-INJECTIVE MODULES

First, we introduce a class of submodule which draws an efficient role in the notion of ic-pseudo-injectivity.

Definition 2.1 *Let M be an R-module. A submodule N of M is called ic-submodules, if N is isomorphic to a closed submodule of M.*

It is clear that, closed submodules (and hence direct summands) are ic-submodules, but the converse generally is not true, nZ is ic-submodule of the Z-module Z which is not closed for each positive integer $n \geq 2$. Every submodule which is isomorphic to ic-submodule in M is itself ic-submodule in M. Every ic-submodule in a closed submodule(direct summand) of M is ic-submodule in M. Let M and N be two isomorphic R-modules. If L is ic-submodule in M, then $\alpha(L)$ is ic-submodule in N where $\alpha : M \to N$ is an isomorphism.

In the following lemma we reformulate continuous modules in terms of ic-submodules, and the proof is immediate.

Lemma 2.2 *An R-module M is continuous if and only if every ic-submodule of M is direct summand.*

In [4], the concept of pri ring R (every right ideal of R is principal) has been generalized to modules. An R-module M is epi-retractable, if every submodule of M is a homomorphic image of M. It was proved the following: Let R be a right hereditary ring. Then R is pri if and only if every free R-module is epi-retractable.

Proposition 2.3 *Let R be a right hereditary pri ring. Then every free R-module is continuous.*

proof: Let M be a free R-module and X an ic-submodule of M. Since R is hereditary, then X is projective. By the above, M is epi-retractable and hence there is an R-epimorphism $\alpha : M \to X$. projectively of X implies that there is an R-homomorphism $\beta : X \to M$ such that $\beta \circ \alpha = I_X$ and hence is a direct summand in M. This shows that M is continuous, lemma(2.2)

In the following, we give a decomposition of projective modules over right hereditary pri rings

Theorem 2.4 *Let R be a right hereditary pri ring. Then every projective R-module is a direct sum of noetherian uniform submodules, each with a division endomorphism ring.*

proof: Let M be a projective R-module. Then there is a free R-module $F = V \oplus U$ such that V is isomorphic to M.Proposition (2.3) implies that F (and hence V) is continuous. Thus M is continuous. Since R is right hereditary, then M is hereditary continuous. By([3],proposition(9)], $M = \oplus_{i \epsilon I} N_i$, where N_i is a noetherian uniform with $End_R(N_i)$ is a division ring for each i.

Definition 2.5 *Let M and N be two R-modules. M is said to be ic-(pseudo)-N-injective, if for each ic-submodule A of N, every R-homomorphism (R-monomorphism) from A to M can be extended to an R-homomorphism from N into M.*

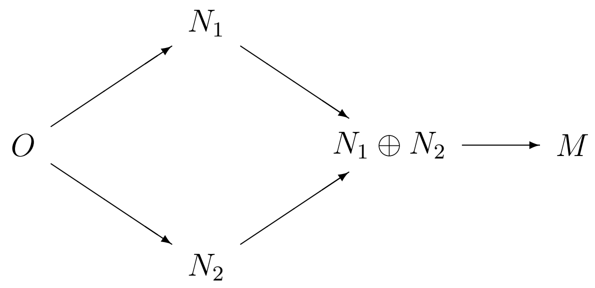
An R-module M is called ic-quasi-injective [11], if M is ic-M-injective. The R-module M is called ic-pseudo-injective, if it is ic-pseudo-M-injective.

Remark 2.6 *1. Every continuous module is ic-pseudo-injective, this follows from the fact that in continuous module, every ic-submodule is direct summand. In particular, every right hereditary pri ring R is ic-pseudo-injective R-module.*

2. *Every pseudo-injective module is ic-pseudo-injective, but the converse may not be true in general. In fact by (1),for example any continuous module is ic-pseudo-injective, but there is continuous modules which are not pseudo-injective, see([2],remark(2.4)).*

3. *Every ic-quasi-injective module is ic-pseudo-injective.*

4. *Let M be a module whose lattice of submodules is*



where N_1 is not isomorphic to N_2, and the endomorphism rings of N_i are isomorphic to $Z/2Z$. The existence of such modules was shown by Hallett and Teply. It was shown in [10], that M is pseudo-injective(and hence ic-pseudi-injective)which is not ic-quasi-injective,since $N_1 \oplus N_2$ is ic-submodule of M and the natural projection of $N_1 \oplus N_2$ onto N_i (i=1,2) can not be extended to an endomorphism of M.

5. *An isomorphic module to ic-pseudo-injective is ic-pseudo-injective.*

Proposition 2.7 *Let M be an R-module and $\{N_i\}_{i \epsilon I}$ a family of R-modules. If $\Pi_{i \epsilon I} N_i$ is ic-pseudo-M-injective , then for each $i \epsilon I$, N_i is ic-pseudo-M-injective.*

proof: Straightforward form the definition and the injections and projections associated with the direct product.

Proposition 2.8 *Let N be an ic-submodule of an R-module M and N be ic-pseudo-M-injective. Then every R-monomorphism from N into$M$ splits. In particular, if M is an R-module whose closed submodules are ic-pseudo-M-injective, then M is CS-module.*

proof: Let $\alpha : N \to M$ be an R-monomorphism. As N is ic-pseudo-M-injective, there is an R-homomorphism $g : M \to M$ which is an extension of $\alpha^{-1} : \alpha(N) \to N$. This shows that $g \circ \alpha = I_N$. Hence $M = \alpha(N) \oplus ker(g)$.

The following is a generalization of Dinh's result in[2].

Proposition 2.9 *Every ic-pseudop-injective module satisfies* (C_2)*.*

proof: Let M be an ic-pseudo-injective R-module , and A a direct summand in M with B is isomorphic to A. Let $f : B \to A$ be an isomorphism. Then A is ic-pseudo-M-injective, proposition (2.7). Whence B is ic-pseudo-M-injective. Proposition (2.8)implies that the R-monomorphism $i_A \circ f : B \to M$ is splits, and hence B is a direct summand in M.

Corollary 2.10 *Let M be an ic-pseudo-injective R-module. Then every submodule of M which is isomorphic to M is a direct summand in M.*

Recall that an R-module M is direct injective , if given any direct summand A of M, an injection $i : A \to M$ and every R-monomorphism $f : A \to M$, there is an R-endomorphism g of M such that $g \circ f = i_A$ [7]. This is equivalent to saying that M satisfies (C_2) [9]. So proposition (2.9) shows that every ic-pseudo-injective module is direct injective. Further, every direct injective module is divisible [5]. Then we have the following:

Proposition 2.11 *The following statements are hold:*

1. *Every ic-pseudo-injective module is divisible.*
2. *Let R be a pri domain. Then an R-module M is ic-pseudo-injective if and only if M is injective.*

In the following, we characterize continuous modules in terms of ic-pseudo-injectivity.

Theorem 2.12 *An R-module M is continuous if and only if every closed submodule of M is ic-pseudo-injective.*

proof: The "only if" part follows from remark (2.6)(1) and proposition(2.7). For the "if" part, the condition (C_2) follows from proposition(2.9), and the CS-module property follows from the particular statement of proposition (2.8).
ic-pseudo-injectivity is not closed under direct sums, as we see in the following example:

Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where $F = Z/2Z, A = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}, B = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$.

It is easy to see that the right R -modules A and B are ic-pseudo-injective (in fact they are quasi-injective). However $R = A \oplus B$ is not ic-pseudo-injective, since otherwise R satisfies (C_2) by proposition (2.9) , but A is isomorphic to C and C is not a direct summand in R .

The concepts of CS-modules and ic-pseudo-injective modules are completely different, the module in remark(2.6)(4) is ic-pseudo-injective which is not CS-module[2], while the Z-module Z is CS-module which is not ic-pseudo-injective.

Now, we explore more properties of ic-pseudo-injective modules.

Theorem 2.13 *If $M_1 \oplus M_2$ is ic-pseudo-injective module, then M_1 and M_2 are mutually ic-injective.*

proof: Let A be an ic-submodule of M_2 and $f : A \to M_1$ be an R-monomorphism. Define $g : A \to M_1 \oplus M_2$ by $g(a) = (f(a), a)$ for all $a \epsilon A$. Then g is an R-monomorphism. Since M_2 is isomprphic to a direct summand of $M_1 \oplus M_2$, then proposition(2.9) implies that M_2 is a direct summand of $M_1 \oplus M_2$. Hence A is ic-submodule of $M_1 \oplus M_2$. So g extends to an endomorphism h of $M_1 \oplus M_2$. Put $h' = h \mid_{M_2}$. If $\pi_1 : M_1 \oplus M_2 \to M_1$ is the natural projection of $M_1 \oplus M_2$ onto M_1, then $\pi_1 \circ h' : M_2 \to M_1$ extends f. This shows that M_1 is ic-pseudo-M_2-injective.

Corollary 2.14 *If $\oplus_{i \epsilon I} M_i$ is ic-pseudo-injective , then M_i is ic-M_j-injective for all $i, j \epsilon I$.*

Corollary 2.15 *For any positive integer $n \geq 2$, if M^n is ic-pseudo-injective, then M is ic-quasi-injective.*

Proposition 2.16 *If M is ic-pseudo-injective R-module, then every submodule of M which is isomorphic to a closed submodule of M is closed.*

proof: Let K be a closed submodule of M and A a submodule of M with an isomorphism $f : A \to K$. Let $i_1 : K \to M$ and $i_2 : A \to M$ be the inclusion mappings. Then there is an endomorphism g of M such that $i_1 \circ f = g \circ i_2$. Pick, by Zorn's lemma, a complement A' in M essentially containing A. The restriction $g \mid_{A'}$ is obviously an R-monomorphism. Hence $K = g(A)$ is essential in $g(A')$, so $A = A'$. This shows that A is closed in M. In the following, we show the converse of the implication mentioned in the introduction under ic-pseudo-injectivity.

Theorem 2.17 *The following statements are equivalent for an R-module M:*

1. *M is continuous.*
2. *M is quasi-continuous and ic-pseudo-injective.*
3. *M is CS-module and ic-pseudo-injective.*

proof: (1) $\Rightarrow$ (2): follows from remark(2.6)(1). (2) $\Rightarrow$ (3): obvious. (3) $\Rightarrow$ (1): follows from proposition(2.9) According to the definition of ic-pseudo-injectivity, every R-monomorphism of ic-submodule of M to M is extendable to all M. In the following we consider a direct sum of ic-submodules instead of individual ic-submodule. For this, we consider the following:
For a positive integer n, consider the following condition for an R-module M.
(W_n): For any submodule K of M such that $K = K_1 \oplus K_2 \oplus ... \oplus K_n$ where K_i is ic-submodule of M for each $i = 1, 2, ..., n$,every R-monomorphism $\alpha K \to M$ can be extended to an R-endomorphism of M.
It is clear that, if M satisfies (W_n), then M satisfies (W_{n-1}) for all $n \geq 2$.

Theorem 2.18 *The following statements are equivalent for a CS-module M:*

1. *M is continuous.*
2. *M satisfies (W_n) for every positive integer n.*
3. *M satisfies (W_n) for every positive integer $n \geq 2$.*
4. *M satisfies (W_2).*
5. *M is ic-pseudo-injective .*

proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5): Clear. (5) $\Rightarrow$ (1): follows from theorem (2.17).

An R-module M is self-similar, if every submodule of M is isomorphic to M [8]. Z as Z-module is self-similar, while $Z/4Z$ as Z-module is not self-similar.

In the following, we show that the distinction between ic-pseudo-injectivity and semisimplcity vanishes for self-similar modules and the proof follows from corollary(2.10)

Proposition 2.19 *Let M be self-similar R-module. Then M is ic-pseudo-injective if and only if M is semisimple.*

It is well-known that an R-module M is injective if and only if M is N-injective for each R-module N, in [6] has been proved that M is injective if and only if is N-pseudo-injective for each R-module N. In the following we get a generalization of these results.

Proposition 2.20 *The following statements are equivalent for an R-module M:*

1. *M is injective .*
2. *M is ic-pseudo- N -injective for each R-module N.*

proof: (1) $\Rightarrow$ (2): Obvious. (2) $\Rightarrow$ (1): Let $E = E(M)$ be the injective hull of M. Then M is ic-submodule in $M \oplus E$. Let $i : M \to E$ be the inclusion mapping and $j : E \to M \oplus E$ the natural injection. ic-pseudo-$M \oplus E$-injectivity of M implies that the identity mapping I_M of M can be extended to an R-homomorphism $f : M \oplus E \to M$. This shows that M is a direct summand of E and hence M is injective.

In the next part we characterize some rings by ic-pseudo-injectivity. Recall that an R-module M is f-injective if M is I-injective for each finitely generated right ideals I.

Theorem 2.21 *A ring R is right noetherian if and only if each f-injective R-module is ic-pseudo-N-injective for each R-module N.*

proof: If R is a right noetherian, then clearly, each f-injective R-module is injective and hence ic-pseudo-N-injective for each R-module N, proposition(2.20). Conversely, assume that every f-injective R-module is ic-pseudo-N-injective for each R-module N. Again proposition(2.20) implies that every f-injective is injective. Let $\{M_i\}_{i \epsilon I}$ be a family of injective R-modules. The $\oplus_{i \epsilon I} M_i$ is f-injective (it is easy to prove), and hence $\oplus_{i \epsilon I} M_i$ is injective. This shows that R is noetherian by using Chase's result in [1].

Proposition 2.22 *The direct sum of any two ic-pseudo-injective modules is ic-pseudo-injective if and only if every ic-pseudo-injective module is injective.*

proof: Let M be an ic-pseudo-injective module, and E its injective hull. ic-pseudo-injectivity of $M \oplus E$ implies that the identity I_M of M extends to an endomorphism g of $M \oplus E$, so M is a direct summand of E and hence $M = E$.The other direction is clear.

Corollary 2.23 *If the direct sum of any two ic-pseudo-injective modules is ic-pseudo-injective, then every ic-pseudo-injective module is quasi-injective.*

Theorem 2.24 *The following statements are equivalent for a ring R.*

1. *R is semisimple artinian.*
2. *Every R-module is ic-pseudo-injective.*
3. *The direct sum of any two ic-pseudo-injective R-modules is ic-pseudo-injective and every cyclic R-module is ic-pseudo-injective.*

proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Clear. (3) $\Rightarrow$ (1): Direct sum of any two ic-pseudo-injective modules is ic-pseudo-injective implies that every ic-pseudo-injective is quasi-injective, corollary (2.23). Then every cyclic R-module is quasi-injective and hence R is semisimple artinian.

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