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Some Remarks on Derivations of Leibniz Algebras

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Abstract

The present paper concerns solvability of finite dimensional Leibniz algebras. As one of the main results of this paper, if Leibniz algebra L has a derivation $d : L \to L$, such that $L^n \subset d(L)$, for some $n > 1$, then L is solvable.

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1 Introduction

In [3] F.Ladish proved that a finite group G, admitting an element a with the property $G' = [a, G]$ is solvable. Using this result M. shahryari in [4] proved a similar theorem for Lie algebras in more general framework, he showed that a finite dimensional Lie algebra L over a field of characteristic zero admitting an abelian algebra of derivations $D \leq Der(L)$, with the following property

$$L^n \subseteq \sum_{d \in D} d(L)$$

for some $n > 1$, is necessarily solvable.
In this work we consider some general properties of Leibniz algebra and its derivation. We extend some results obtained for derivations of Lie algebras in [4] to the case of Leibniz algebras.
It is worth noting that in 1955, Jacobson [2] proved essential theorem in which every Lie algebra over a field of characteristic zero admitting a nonsingular derivation is nilpotent.

2 PRELIMINARIES

In this section we give necessary definitions and preliminary concepts.

Definition 2.1 *An algebra $(L, [-,-])$ over a field F is said to be a Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity*

$$\big[x, [y, z]\big] = \big[[x, y], z\big] - \big[[x, z], y\big]$$

holds.

A subalgebra H of a Leibniz algebra is said to be a two-sided ideal if $[L, H] \subseteq L$ and $[H, L] \subseteq L$. Let H and K be two sided ideals of L. The commutator ideal of H and K, denoted by $[H, K]$ is the two-sided ideal of L spanned by the brackets $[h, k], [k, h], h \in H, k \in K$.
For a given Leibniz algebra L, we define the lower central and derived series to the sequences of two-sided ideals defined recursively as follows:

$$L^1 = L,\ L^{k+1} = [L^k, L],\ k \geq 1\ ;\ L^{[1]} = L,\ L^{[s+1]} = \big[L^{[s]}, L^{[s]}\big],\ s \geq 1.$$

Definition 2.2 *Leibniz algebra L is said to be solvable (nilportent) if there exists $n \in \mathbb{N}$ ($m \in \mathbb{N}$) such that $L^{[n]} = 0$ ($L^m = 0$).*

Definition 2.3 *For a Leibniz algebra L, a Linear map $d : L \to L$ is said to be a derivation if*

$$d[x, y] = [d(x), y] + [x, d(y)]$$

for all $x, y \in L$.

Definition 2.4 *Inner derivations are a kind of derivations in which we have $R_x : L \to L$ (for a fixed $x \in L$) such that $R(x)(y) = [y, x]$ is a derivation.*

Definition 2.5 *The form $f(a, b) = tr(R_a, R_b)$ for $a, b \in L$ is called the killing form of the Leibniz algebra L. A biLinear form (a, b) on L satisfying the condition*

$$f([a, c], b) + f(a, [b, c]) = 0$$

is called an invariant form on L.

Theorem 2.6 *Let L be a Leibniz algebra over an algebraically closed field of zero characteristic. Then L is solvable if and only if*
$tr(R_a, R_b) = 0$ for any $a \in L^2$.

proof. [1]

3 Main theorems and Results

In this section L is finite dimensional Leibniz algebra over a field K of characteristic zero. L^n and $L^{[n]}$ represent the n-th terms of lower central series and derived series of L, respectively. Also we denote by $Der(L)$ the algebra of derivation of L.

Definition 3.1 *A subspace $h \subset L$ is called left (resp.right) ideal if for any $a \in h$ and $x \in L$ one has $[x, a] \in h$ (resp. $[a, x] \in h$), if h is both left and right ideal, then h is called two-sided ideal.*

Theorem 3.2 *Let L be a finite dimensional Leibniz algebra over K and suppose that U is a two-sided ideal. Suppose that there is an abelian subalgebra A such that $U^n \subseteq [A, U]$, for some $n > 1$. Then U is solvable.*

proof. We assume that K is algebraically closed, so we use Cartan's criterion. Let $S = U^n$ and define a bilinear form on L by

$$f(\alpha, \beta) = tr(R_\alpha, R_\beta) \quad for \alpha, \beta \in L$$

f is killing form of L. Since S is an ideal of L, f is associative,

$$f([a, b], c) = f(a, [b, c]).$$

Now we restrict f on S, so we have

$$f_S(a, b) = tr(R_a, R_b) \quad for a, b \in S$$

We apply Cartan's criterion for algebra $R(S)$ and consider as $R(S)$-module. If we consider $R(S) = \frac{S}{Z(S)}$ where $Z(S)$ is the center of S. Obviously $f(S, S') = 0$ implies $f(S', S') = 0$, that is $tr(R_a, R_b) = 0$ for any $a \in S$. Thus $R(S)$ is solvable and solvability of $R(S)$ is equivalent to solvability of S. Hence U is solvable.

We now suppose that K is not necessarily algebraically closed, so we use the algebraic closure of K. Let $\bar{K}$ be algebraic closure of K and $\bar{L} = \bar{K} \otimes_K A$. Now $\bar{L}$ is a finite dimensional Leibniz algebra over $\bar{K}$ in which $\bar{K} \otimes_K U$ is an ideal and $\bar{K} \otimes_u U$ is an abelian subalgebra. Further, we have

$$(\bar{K} \otimes_K U)^n = \bar{K} \otimes_K U^n \subseteq [\bar{K} \otimes_K A, \bar{K} \otimes_K U].$$

So $\bar{K} \otimes_K U$is solvable, that is, there exist a number m such that $(\bar{K} \otimes_K U)^{(m)} = 0$. On the other hand,

$$(\bar{K} \otimes_K U)^{(m)} = \bar{K} \otimes_K U^{(m)},$$

therefore $U^{(m)} = 0$.

Corollary 3.3 *Suppose that there exist an abelian subalgebra $A \leq L$ and an integer $n > 1$, such that $L^n \subseteq [A, L]$. Then L is solvable.*

Finally we apply Theorem 3.2 to derivations of Leibniz algebra.

Corollary 3.4 *Suppose that there exist an abelian subalgebra $D \leq Der(L)$ and an integer $n > 1$, such that*

$$L^n \subseteq \sum_{d \in D} d(L).$$

Then L is solvable.

Proof. Let $\hat{L} = D \ltimes L$, the natural semidirect product. We assume that L is two-sided ideal. D is an abelian subalgebra in $\hat{L}$. Note that in the semidirect product,

$$[D, L] = \sum_{d \in D} d(L),$$

hence the assumption is just $L^n \subseteq [D, L]$. So L is solvable by Theorem 3.2.

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