International Journal of Algebra, Vol. 6, 2012, no. 30, 1463 - 1470

On Sandwich Sets of Elements in Regular Γ-Semigroups

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Abstract

The aim of this paper is to introduce an (α, β, θ)-sandwich set of idempotent elements and study some properties of an (α, β, θ)-sandwich set of elements in regular Γ-semigroups. We determine when an (α, β, θ)-sandwich set of elements is a right θ-zero semigroup.

Mathematics Subject Classification: 20M10

Keywords: (α, β, θ)-sandwich element, right θ-zero semigroup

1 Introduction

The concept of sandwich sets $S(e, f)$ in regular semigroups was first introduced by Nambooripad [7]. This concept was used in describing the structure of regular semigroups. After that, the concept has been widely used to study the properties of regular semigroups [8], [3]. Zhu and He [15] used the sandwich set $S(e, f)$ to show that the sandwich set on some special classes of semigroups has only one element. Petrich [9] studied characterization of one-sided sandwich sets and studied sandwich sets of elements in regular semigroups that are right zero semigroups. The other papers about sandwich sets in semigroups can be seen in [15].

The concept of Γ-semigroups has been studied by Sen [11] in 1981. Sen and Saha [10] changed the definition, which is more general definition and gave the definition of the Γ-semigroup via a mapping as follows: A nonempty set S is called a Γ-*semigroup* if there exists a mapping from $S \times \Gamma \times S$ to S written as $(a, \alpha, b) \mapsto a\alpha b$ satisfying the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. In this paper, we introduce an (α, β, θ)-sandwich set of idempotents $E(S)$ and study some properties of an (α, β, θ)-sandwich set of elements in regular Γ-semigroups. We determine when an (α, β, θ)-sandwich set of elements is a right θ-zero semigroup and it has only one element.

2 Preliminary

An element e of a regular Γ-semigroup S is called an α-*idempotent* [14], where $\alpha \in \Gamma$, if $e\alpha e = e$. The set of all α-idempotents is denoted by $E_\alpha(S)$ and the The set $\bigcup_{\alpha\in\Gamma} E_\alpha(S)$, set of all α-idempotents for all $\alpha \in \Gamma$, is denoted by $E(S)$. Every element of $E(S)$ is called an *idempotent element* of S. An element a in a Γ-semigroup S is *regular* [12] if there exist $y \in S$, $\alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x$. A Γ-semigroup S is said to be a *regular* Γ-*semigroup* [12] if every element in the Γ-semigroup is regular. For any a Γ-semigroup S, $a, x \in S$ and $\alpha, \beta \in \Gamma$, if $a = a\alpha x\beta a$ and $x = x\beta a\alpha x$ then x is called an (α, β)-*inverse of* a [12]. Note that, if $a = a\alpha x\beta a$ for some $\alpha, \beta \in \Gamma$ then $a\alpha x \in E_\beta(S)$ and $x\beta a \in E_\alpha(S)$. The set of all (α, β)-inverses of an element a in a Γ-semigroup S is denoted by $V_\alpha^\beta(a)$. That is,

$$V_\alpha^\beta(a) := \{x \in S \mid a = a\alpha x\beta a \text{ and } x = x\beta a\alpha x\}.$$

If a is a regular element then $V_\alpha^\beta(a) \neq \emptyset$ for some $\alpha, \beta \in \Gamma$ [14]. In a regular Γ-semigroup S, we have that $E(S)$ is a non-empty set.

Green's equivalence relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ and $\mathcal{D}$ on a Γ-semigroup S were studied by Chinram and Siammai [1] and showed that:

Lemma 2.1. [1] *Let S be a Γ-semigroup. Then for all $a, b \in S$,*
(1) $a\mathcal{L}b$ if and only if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = x\alpha b$ and $b = y\beta a$,
(2) $a\mathcal{R}b$ if and only if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = b\alpha x$ and $b = a\beta y$,
(3) $a\mathcal{H}b$ if and only if $a\mathcal{L}b$ and $a\mathcal{R}b$.

Lemma 2.2. [1] *Let S be a Γ-semigroup, $\alpha \in \Gamma$ and e be an α-idempotent. Then*
(1) $a\alpha e = a$ for all $a \in L_e$,
(2) $e\alpha a = a$ for all $a \in R_e$,
(3) $a\alpha e = a = e\alpha a$ for all $a \in H_e$.

3 Main Results

In this section, we introduce an (α, β, θ)-sandwich set of idempotents e and f as follow: For a Γ-semigroup $S, \alpha, \beta, \theta \in \Gamma$ and $e \in E_\alpha(S), f \in E_\beta(S)$, we define a set $S_\theta^{(\alpha,\beta)}(e, f)$ by

$$S_\theta^{(\alpha,\beta)}(e, f) := \{g \in V_\beta^\alpha(e\theta f) \cap E_\theta(S) \mid g\alpha e = f\beta g = g\}.$$

Then $S_\theta^{(\alpha,\beta)}(e, f)$ may be an empty set. If $S_\theta^{(\alpha,\beta)}(e, f) \neq \emptyset$ then $S_\theta^{(\alpha,\beta)}(e, f)$ is called an (α, β, θ)-*sandwich set* of idempotents e and f. It is easy to verify that if S is a regular Γ-semigroup then $S_\theta^{(\alpha,\beta)}(e, f) \neq \emptyset$. From the definition, we have

Proposition 3.1. [2] *Let S be a regular Γ-semigroup, $\alpha, \beta, \theta \in \Gamma$ and $e \in E_\alpha(S), f \in E_\beta(S)$. Then*

$$S_\theta^{(\alpha,\beta)}(e, f) = \{g \in E_\theta(S) \mid g\alpha e = f\beta g = g \text{ and } e\theta g\theta f = e\theta f\}.$$

Proposition 3.2. [2] *Let S be a regular Γ-semigroup and $a \in S$. Then there exist $\alpha, \beta \in \Gamma, a' \in V_\alpha^\beta(a)$ such that*

$$S_\theta^{(\alpha,\beta)}(a'\beta a, a\alpha a') = a\alpha V_\alpha^\beta(a\theta a)\beta a$$

for all $\theta \in \Gamma$.
Furthermore, if $a'' \in V_\alpha^\beta(a)$ then $S_\theta^{(\alpha,\beta)}(a''\beta a, a\alpha a'') = a\alpha V_\alpha^\beta(a\theta a)\beta a$.

In Proposition 3.2, we see that $S_\theta^{(\alpha,\beta)}(a'\beta a, a\alpha a') = a\alpha V_\alpha^\beta(a\theta a)\beta a$ for all choice $a' \in V_\alpha^\beta(a)$.
For $a \in S$ and $\alpha, \beta \in \Gamma$, define

$$S_\theta^{(\alpha,\beta)}(a) := S_\theta^{(\alpha,\beta)}(a'\beta a, a\alpha a').$$

Then $S_\theta^{(\alpha,\beta)}(a)$ is called an (α, β, θ)-*sandwich set of an element a.*

Lemma 3.3. [2] *Let e be an α-idempotent and b an β-idempotent of a regular Γ-semigroup S. If e and f are $\mathcal{D}$-related then there exist $a \in S$ and $a' \in V_\alpha^\beta(a)$ such that $e = a'\beta a$ and $f = a\alpha a'$.*

We use the definition of the natural partial order on a regular Γ-semigroup in next proposition.

Proposition 3.4. [2] *Let a and b be elements of a regular Γ-semigroup S. Then $a \leqslant b$ if and only if there exist $\beta, \gamma \in \Gamma, f \in E_\beta(S), g \in E_\gamma(S)$ such that $a = f\beta b = b\gamma g$.*

Lemma 3.5. [2] *Let S be a regular Γ-semigroup and $\alpha, \beta, \theta \in \Gamma$. If $e \in E_\alpha(S)$, $f \in E_\beta(S)$ and $x \in V_\alpha^\beta(e\theta f)$ for any $\theta \in \Gamma$ then*

$$S_\theta^{(\alpha,\beta)}(e,f)\theta f = S_\beta^{(\alpha,\beta)}(x\beta e\theta f, f) = \{q \in S \mid e\theta q = e\theta f \mathcal{L} q \leqslant f\}$$

and

$$e\theta S_\theta^{(\alpha,\beta)}(e,f) = S_\alpha^{(\alpha,\beta)}(e, e\theta f\alpha x) = \{r \in S \mid r\theta f = e\theta f \mathcal{R} r \leq e\}.$$

For $\theta \in \Gamma$, a Γ-semigroup S is called a *right (left) θ-zero semigroup* if $a\theta b = b$ ($a\theta b = a$) for all $a, b \in S$.

We shall give some necessary and sufficient conditions for a right θ-zero semigroup $S_\theta^{(\alpha,\beta)}(a)$.

Theorem 3.6. *Let S be a regular Γ-semigroup and $\alpha, \beta, \theta \in \Gamma$, the following conditions are equivalent.*

(1) For any $a \in S$, $S_\theta^{(\alpha,\beta)}(a)$ is a right θ-zero semigroup.

(2) If $e \in E_\alpha(S), f \in E_\beta(S)$ such that $e\mathcal{D}f$ then $S_\theta^{(\alpha,\beta)}(e,f)$ is a right θ-zero semigroup.

(3) If $a \in S$ and $x, y \in V_\alpha^\beta(a\theta a)$ then $(a\alpha x\beta a)\theta(a\alpha y\beta a) = a\alpha y\beta a$.

(4) If $a, x, y \in S$ with $a\theta a = a\theta a\alpha x\beta a\theta a = a\theta a\alpha y\beta a\theta a$ then
$(a\alpha x\beta a)\theta(a\alpha y\beta a) = (a\alpha y\beta a)\theta(a\alpha y\beta a)$.

(5) If $e \in E_\alpha(S), f \in E_\beta(S)$ such that $e\mathcal{D}f$ and $x, y \in S$,
$e\theta x = e\theta y = e\theta f\mathcal{L}x\mathcal{L}y, \quad x, y \leqslant f$ *then $x = y$.*

Proof. (1) $\Rightarrow$ (2) Let $e \in E_\alpha(S), f \in E_\beta(S)$ such that $e\mathcal{D}f$. By Lemma 3.3, there exist $a \in S$ and $a' \in V_\alpha^\beta(a)$ such that $e = a'\beta a$ and $f = a\alpha a'$. Thus

$$S_\theta^{(\alpha,\beta)}(e,f) = S_\theta^{(\alpha,\beta)}(a'\beta a, a\alpha a') = S_\theta^{(\alpha,\beta)}(a)$$

is a right θ-zero semigroup.

(2) $\Rightarrow$ (3) Let $a \in S, \theta \in \Gamma$ and $a' \in V_\alpha^\beta(a)$. Then $a'\beta a\mathcal{L}a$ and $a\mathcal{R}a\alpha a'$, so $a'\beta a\mathcal{D}a\alpha a'$. By assumption, we have that $S_\theta^{(\alpha,\beta)}(a)$ is a right θ-zero semigroup. Let $x, y \in V_\alpha^\beta(a\theta a)$. By Proposition 3.2, we obtain $a\alpha x\beta a, a\alpha y\beta a \in S_\theta^{(\alpha,\beta)}(a)$. Thus $(a\alpha x\beta a)\theta(a\alpha y\beta a) = a\alpha y\beta a$.

(3) $\Rightarrow$ (4) Let $a, x, y \in S$ be such that $a\theta a = a\theta a\alpha x\beta a\theta a = a\theta a\alpha y\beta a\theta a$. Then $x\beta a\theta a\alpha x, y\beta a\theta a\alpha y \in V_\alpha^\beta(a\theta a)$. By hypothesis, we have

$$[a\alpha(x\beta a\theta a\alpha x)\beta a]\theta[a\alpha(y\beta a\theta a\alpha y)\beta a] = a\alpha(y\beta a\theta a\alpha y)\beta a.$$

Thus

$$\begin{aligned}(a\alpha y\beta a)\theta(a\alpha y\beta a) &= a\alpha x\beta a\theta a\alpha x\beta a\theta a\alpha y\beta a\theta a\alpha y\beta a \\ &= a\alpha x\beta a\theta a\alpha y\beta a\theta a\alpha y\beta a \\ &= a\alpha x\beta a\theta a\alpha y\beta a.\end{aligned}$$

(4) ⇒ (5) Let $e \in E_\alpha(S)$ and $f \in E_\beta(S)$ be such that $e\mathcal{D}f$. By Lemma 3.3, there exist $a \in S$ and $a' \in V_\alpha^\beta(a)$ such that $e = a'\beta a$ and $f = a\alpha a'$. By Proposition 3.2, we have

$$a\alpha V_\alpha^\beta(a\theta a)\beta a = S_\theta^{(\alpha,\beta)}(a) = S_\theta^{(\alpha,\beta)}(e,f).$$

Let $x, y \in S$ be such that $e\theta x = e\theta y = e\theta f\mathcal{L}x\mathcal{L}y$ and $x, y \leqslant f$. By Lemma 3.5, we have $x, y \in S_\theta^{(\alpha,\beta)}(e,f)\theta f$. Then $x, y \in a\alpha V_\alpha^\beta(a\theta a)\beta a\theta f$. Thus there exist $s, t \in V_\alpha^\beta(a\theta a)$ such that $x = a\alpha s\beta a\theta f$ and $y = a\alpha t\beta a\theta f$. Since $s, t \in V_\alpha^\beta(a\theta a)$, we have $a\theta a = a\theta a\alpha s\beta a\theta a = a\theta a\alpha t\beta a\theta a$. By hypothesis, we have $(a\alpha s\beta a)\theta(a\alpha t\beta a) = (a\alpha t\beta a)\theta(a\alpha t\beta a) = a\alpha t\beta a$. It follows that

$$x = a\alpha s\beta a\alpha a'\beta a\theta f = a\alpha s\beta a\alpha a'\beta a\theta a\alpha t\beta a\theta a\alpha a' = a\alpha t\beta a\theta a\alpha a' = y.$$

(5) ⇒ (1) Let $a \in S$ and $a' \in V_\alpha^\beta(a)$. Set $e = a'\beta a$ and $f = a\alpha a'$. We will show that $S_\theta^{(\alpha,\beta)}(e,f)\theta f$ is trivial. Let $x, y \in S_\theta^{(\alpha,\beta)}(e,f)\theta f$. By Lemma 3.5, we have $e\theta x = e\theta f\mathcal{L}x \leqslant f$ and $e\theta y = e\theta f\mathcal{L}y \leqslant f$, so $e\theta x = e\theta y = e\theta f\mathcal{L}x\mathcal{L}y \leqslant f$. Since $e = a'\beta a$ and $a = a\alpha a'\beta a = a\alpha e$, we get that $e\mathcal{L}a$. And since $f = a\alpha a'$ and $a = a\alpha a'\beta a = f\beta a$, we obtain $a\mathcal{R}f$. Thus $\mathcal{D}f$. By hypothesis, we have $x = y$. Then $S_\theta^{(\alpha,\beta)}(e,f)\theta f$ is trivial. Therefore $S_\theta^{(\alpha,\beta)}(a)$ is a right θ-zero semigroup. □

Let S and S' be Γ-semigroups and $\theta \in \Gamma$. The mapping $\varphi : S \to S'$ is called an *θ-homomorphism* if $(a\theta b)\varphi = (a\varphi)\theta(b\varphi)$ for all $a, b \in S$. Let φ be an θ-homomorphism of S into S' and let ψ be an θ-homomorphism of S' into S. If $\psi \circ \varphi$ is the identity mapping of S onto itself, and if $\varphi \circ \psi$ is the identity mapping of S' onto itself, then φ is an *θ-isomorphism* of S onto S', and ψ is the *inverse θ-isomorphism*. Such θ-isomorphisms ψ and φ are called *mutually inverse θ-isomorphisms*. The next theorem is very important results for the main theorem.

Theorem 3.7. [2] *Let S be a regular Γ-semigroup and $\alpha, \beta, \theta \in \Gamma$. For $e \in E_\alpha(S), f \in E_\beta(S)$, the mappings*

$$\varphi : x \to (x\theta f, e\theta x), \quad \psi : (y, z) \to y\alpha w\beta z$$

(where $w \in V_\alpha^\beta(e\theta f)$) are mutually inverse θ-isomorphisms between sub Γ-semigroup $S_\theta^{(\alpha,\beta)}(e,f)$ and $S_\theta^{(\alpha,\beta)}(e,f)\theta f \times e\theta S_\theta^{(\alpha,\beta)}(e,f)$.

Proposition 3.8. *Let e be α-idempotent and f be β-idempotent and $\theta \in \Gamma$. If $p, q \in S_\theta^{(\alpha,\beta)}(e,f)$ then $p = p\theta q\theta p$.*

Proof. Let $p, q \in S_\theta^{(\alpha,\beta)}(e,f)$. Then $p\theta q\theta p = p\alpha e\theta f\beta p = p$. □

The main result show that an (α, β, θ)-sandwich set $S_\theta^{(\alpha,\beta)}(a)$ has only one element and we denote the cardinality of a set X by $\mid X \mid$.

Theorem 3.9. *Let S be a regular Γ-semigroup and $\alpha, \beta, \theta \in \Gamma$, the following statements are equivalent.*

(1) For any $a \in S$, $\mid S_\theta^{(\alpha,\beta)}(a) \mid = 1$.

(2) If $a \in S$ and $x, y \in V_\alpha^\beta(a\theta a)$ then $a\alpha x\beta a = a\alpha y\beta a$.

(3) If $a, x, y \in S$ with $a\theta a = a\theta a\alpha x\beta a\theta a = a\theta a\alpha y\beta a\theta a$ then $(a\alpha x\beta a)\theta(a\alpha x\beta a) = (a\alpha y\beta a)\theta(a\alpha y\beta a)$.

(4) If $a, x, y \in S$ with $a\theta a = a\theta a\alpha x\beta a\theta a = a\theta a\alpha y\beta a\theta a$ then $(a\alpha x\beta a)\theta(a\alpha y\beta a) = (a\alpha y\beta a)\theta(a\alpha x\beta a)$.

(5) If $e \in E_\alpha(S), f \in E_\beta(S)$ such that $e\mathcal{D}f$ then $\mid S_\theta^{(\alpha,\beta)}(e, f) \mid = 1$.

Proof. (1) $\Rightarrow$ (2) Let $a \in S$ and $x, y \in V_\alpha^\beta(a\theta a)$. By Corollary 3.2, we have $S_\theta^{(\alpha,\beta)}(a) = a\alpha V_\alpha^\beta(a\theta a)\beta a$. By hypothesis, we have $a\alpha x\beta a = a\alpha y\beta a$.

(2) $\Rightarrow$ (3) Let $a, x, y \in S$ be such that $a\theta a = a\theta a\alpha x\beta a\theta a = a\theta a\alpha y\beta a\theta a$. Then $x\beta(a\theta a)\alpha x, y\beta(a\theta a)\alpha y \in V_\alpha^\beta(a\theta a)$. By hypothesis, we have

$$a\alpha(x\beta a\theta a\alpha x)\beta a = a\alpha(y\beta a\theta a\alpha y)\beta a.$$

(3) $\Rightarrow$ (4) Let $a, x, y \in S$ be such that $a\theta a = a\theta a\alpha x\beta a\theta a = a\theta a\alpha y\beta a\theta a$. By the hypothesis, we have

$$\begin{aligned}(a\alpha x\beta a)\theta(a\alpha y\beta a) &= (a\alpha x\beta a)\theta(a\alpha x\beta a)\theta(a\alpha y\beta a)\\ &= (a\alpha y\beta a)\theta(a\alpha y\beta a)\theta(a\alpha y\beta a)\\ &= (a\alpha y\beta a)\theta(a\alpha x\beta a)\theta(a\alpha x\beta a)\\ &= a\alpha y\beta a\theta a\alpha x\beta a.\end{aligned}$$

(4) $\Rightarrow$ (5) Let $e \in E_\alpha(S), f \in E_\beta(S)$ be such that $e\mathcal{D}f$.By Lemma 3.3, there exist $a \in S$ and $a' \in V_\alpha^\beta(a)$ such that $e = a'\beta a$ and $f = a\alpha a'$. Let $x, y \in S_\theta^{(\alpha,\beta)}(a'\beta a, a\alpha a')$. Then $a\alpha a'\beta x\alpha a'\beta a = x$ and $a\alpha a'\beta y\alpha a'\beta a = y$. Thus $a\theta x\theta a = a\theta a$. Similarly, we can show that $a\theta y\theta a = a\theta a$. Indeed,

$$(a\theta a)\alpha(a'\beta x\alpha a')\beta(a\theta a) = a\theta a = (a\theta a)\alpha(a'\beta y\alpha a')\beta(a\theta a).$$

By hypothesis, we have that

$$[a\alpha(a'\beta x\alpha a')\beta a]\theta[a\alpha(a'\beta y\alpha a')\beta a] = [a\alpha(a'\beta y\alpha a')\beta a]\theta[a\alpha(a'\beta x\alpha a')\beta a]$$

which implies that $x\theta y = y\theta x$. By Proposition 3.8, we get that

$$x = x\theta y\theta x = x\theta x\theta y = x\theta y = x\theta y\theta y = y\theta x\theta y = y.$$

(5) $\Rightarrow$ (1) Let $a \in S$. Since S is a regular Γ-semigroup, there exist $\alpha, \beta \in \Gamma$ such that $a' \in V_\alpha^\beta(a)$. Set $e := a'\beta a$ and $f := a\alpha a'$. Then it is easy to show that $e\mathcal{D}f$. By hypothesis, we obtain that $\mid S_\theta^{(\alpha,\beta)}(a) \mid = 1$. □

Corollary 3.10. *Let S be a regular Γ-semigroup and $\alpha, \beta, \theta \in \Gamma$, the following conditions are equivalent.*

(1) If $e \in E_\alpha(S), f \in E_\beta(S)$ such that $e\mathcal{D}f$ then $\mid S_\theta^{(\alpha,\beta)}(e,f) \mid = 1$.

(2) For any and $x, y \in S$,

(i) if $e\theta x = e\theta y = e\theta f \mathcal{L} x \mathcal{L} y$ and $x, y \leqslant f$ then $x = y$,

(ii) if $x\theta f = y\theta f = e\theta f \mathcal{R} x \mathcal{R} y$ and $x, y \leqslant e$ then $x = y$.

Proof. (1) $\Rightarrow$ (2) Let $e \in E_\alpha(S), f \in E_\beta(S)$ be such that $e\mathcal{D}f$ and $x, y \in S$, $e\theta x = e\theta y = e\theta f \mathcal{L} x \mathcal{L} y$ and $x, y \leqslant f$. By Lemma 3.5, we have $x, y \in S_\theta^{(\alpha,\beta)}(e,f)\theta f$. Then there exist $p, q \in S_\theta^{(\alpha,\beta)}(e,f)$ such that $x = p\theta f$ and $y = q\theta f$. By assumption, $p = q$ which implies that $x = p\theta f = q\theta f = y$. Similarly, we can show that if $x\theta f = y\theta f = e\theta f \mathcal{R} x \mathcal{R} y$ and $x, y \leqslant e$. Then $x = y$.

(2) $\Rightarrow$ (1) Let $a \in S$. Set $e = a'\beta a$ and $f = a\alpha a'$. Then it is easy to show that $e\mathcal{D}f$. Claim that $\mid S_\theta^{(\alpha,\beta)}(e,f)\theta f \mid = 1$ and $\mid e\theta S_\theta^{(\alpha,\beta)}(e,f) \mid = 1$. Let $x, y \in S_\theta^{(\alpha,\beta)}(e,f)\theta f$. Then $e\theta x = e\theta f \mathcal{L} x \leqslant f$ and $e\theta y = e\theta f \mathcal{L} y \leqslant f$. Thus $e\theta x = e\theta y = e\theta f \mathcal{L} x \mathcal{L} y$ and $x, y \leqslant f$. By assumption, we have $x = y$. Hence $S_\theta^{(\alpha,\beta)}(e,f)\theta f$ is trivial. Similarly, we can show that $e\theta S_\theta^{(\alpha,\beta)}(e,f)$ is trivial. By Theorem 3.7, $S_\theta^{(\alpha,\beta)}(e,f) \cong_\theta S_\theta^{(\alpha,\beta)}(e,f)\theta f \times e\theta S_\theta^{(\alpha,\beta)}(e,f)$. Thus $\mid S_\theta^{(\alpha,\beta)}(e,f) \mid = 1$.

□

Corollary 3.11. *Let S be a regular Γ-semigroup and $\alpha, \beta, \theta \in \Gamma$. For any $e \in E_\alpha(S), f \in E_\beta(S)$ the following conditions are equivalent.*

(1) $S_\theta^{(\alpha,\beta)}(e,f)$ is a right zero θ-semigroup.

(2) $\mid S_\theta^{(\alpha,\beta)}(e,f)\theta f \mid = 1$.

(3) $\mid S_\beta^{(\alpha,\beta)}(x\beta e\theta f, f) \mid = 1$ for any $x \in V_\alpha^\beta(e\theta f)$.

Proof. (1) $\Rightarrow$ (2) Let $x, y \in S_\theta^{(\alpha,\beta)}(e,f)\theta f$. Then $x = s\theta f, y = t\theta f$ for some $s, t \in S_\theta^{(\alpha,\beta)}(e,f)$. Indeed,

$$x = s\alpha e\theta t\theta f = s\theta t\theta f = y.$$

Thus $\mid S_\theta^{(\alpha,\beta)}(e,f)\theta f \mid = 1$.

(2) $\Rightarrow$ (1) Suppose that $S_\theta^{(\alpha,\beta)}(e,f)\theta f$ is trivial. Let $x, y \in S_\theta^{(\alpha,\beta)}(e,f)$. Then $x\theta f = y\theta f$ which implies that $x\theta y = x\alpha e\theta f\beta y = y\theta f\beta y = y$. Therefore $S_\theta^{(\alpha,\beta)}(e,f)$ is a right zero θ-semigroup.

(2) $\Leftrightarrow$ (3) It follows from Lemma 3.5. □

ACKNOWLEDGEMENTS. This work is supported by Office of the Higher Education Commission (OHEC), Thailand.

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Received: October, 2012