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# Crossing Number of a Zero Divisor Graph

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**Abstract**

Let $R$ be a commutative ring and let $\Gamma(Z_n)$ be the zero divisor graph of a commutative ring R, whose vertices are non-zero zero divisors of $Z_n$, and such that the two vertices $u$, $v$ are adjacent if n divides uv. In this paper, we evaluate the crossing number of $\Gamma(Z_n)$ for some cases of n. Finally we posted two open conjectures, i) for any graph $\Gamma(Z_{pq})$, where p and q are distinct prime numbers with $p < q$ then cr $(\Gamma(Z_{pq})) = (p-1)(p-3)(q-1)(q-3)/16$, ii) for any graph $\Gamma(Z_{p^2})$, where p is any prime then $cr(\Gamma(Z_{p^2})) = (p-1)(p-3)^2(p-5)/64$.

**Mathematics Subject Classification:** 05C25

**Keywords:** Commutative ring, Zero divisor graph, Crossing number

# 1 Introduction

Let R be a commutative ring and let Z(R) be its set of zero-divisors. We associate a graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) - \{0\}$, the set of non-zero zero divisors of R and for distinct $u, v \in Z(R)^*$, the vertices $u$ and $v$

are adjacent if and only if $uv = 0$. Throughout this paper, the commutative ring R is $Z_n$ and zero divisor graph $\Gamma(R)$ is $\Gamma(Z_n)$. The idea of a zero-divisor graph of a commutative ring was introduced by I. Beck in [2], where he was mainly interested in colorings. For notation and graph theory terminology, are considered as in [1, 8] and the basic for commutative ring theory as in [4].

The crossing number cr(G) of a graph G is the minimum number of edge crossings among the drawings of G in the plane. A drawing of a graph is "good" if and only if all edges intersect at most once. A good drawing of a graph G satisfies the following; i) adjacent edges never cross, ii) two non-adjacent edges cross at most once, iii) no more than two edges cross at a point of the plane, and iv) no edge passes through a vertex of the graph G[5].

Turan's brick factory problem asks the question of finding the crossing number of a complete bipartite graph. Zarankiewicz [9] proposed a solution to the problem in 1953. Guy [3] points out an error in Zarankiewicz's proof that was discovered in 1965 by Kainen and Ringel. The error was the assumption that among the $m$ graphs $K_{1,m}$ that compose $K_{m,n}$ it is always possible to find two which do not contain a crossing. To date, Zarankiewicz's conjecture has not been proven or disproven and stands as an upper bound.

## 2 Crossing Number of a Zero Divisor Graph

In this section, we evaluate the crossing number of some zero divisor graphs in a commutative ring.

**Lemma 2.1** *A graph $\Gamma(Z_n)$ is connected if and only if $n$ is a composite number*[7].

**Theorem 2.2** *In $\Gamma(Z_{2p})$, where $p$ is any prime $> 2$, then $cr(\Gamma(Z_{2p})) = 0$.*

***Proof:*** *The vertex set of $\Gamma(Z_{2p})$ is {2,4,...,2(p-1),p}. Let $u = 2(p-1)$ and $v = p$ then $uv = 2(p-1)p = 2p(p-1)$. Clearly, $2p$ must divides $2p(p-1)$, then there exist a edge connect between $u$ and $v$. Similarly, let $x$ be any vertex in $\{2, 4, ..., 2(p-1)\}$ and $v = p$ then $2p$ must divides $xv$. Note that, $v$ is adjacent to all the vertices in $\Gamma(Z_{2p})$ and we know that $\Gamma(Z_{2p})$ is a star graph*[7]*, which implies that $cr(\Gamma(Z_{2p})) = 0$.*

**Theorem 2.3** *For any graph $\Gamma(Z_{3p})$, where $p$ is any prime $> 3$, then $cr(\Gamma(Z_{3p})) = 0$.*

***Proof:*** *The vertex set of $\Gamma(Z_{3p})$ is $\{3, 6, 9, ..., 3(p-1), p, 2p\}$. Let $u$ and $v$ be two vertices in $\Gamma(Z_{3p})$ with maximum degree. Let $u = p$ and $v = 2p$ then there exist any other vertex $w \neq p \neq 2p$ in $\Gamma(Z_{3p})$ such that $w$ is adjacent to both $u$ and $v$. That is, $uw = vw = 0$. But $uv = 2p^2$ which is not divided by $3p$. Therefore $u$ and $v$ are non-adjacent vertices. Then the vertex set $V$*

*can be partitioned into two parts* $V_1$ *and* $V_2$ *such that* $V_1 = \{u, v\} = \{p, 2p\}$ *and* $V_2 = \{3, 6, 9, ...., 3(p-1)\}$. *Clearly* $|V_1| = 2$ *and* $|V_2| = p - 1$, *then* $|V| = |V_1 + |V_2| = p + 1$.

*Let the elements in* $V_1$ *be* $v_1, v_2$ *and in* $V_2$ *be* $u_1, u_2, ...u_{p-1}$. *In, drawing D, we denote by* $cr_D(v_i, u_j)$ *the number of crossing arcs, one terminating at* $v_i$ *and the other at* $u_j$. *Clearly, the number of crossing arcs which terminate at* $v_i$ *is,*

$$cr_D(v_i) = \sum_{j=1}^{p-1} cr_D(v_i, u_j) \text{ for } i = 1, 2.$$

*which implies that*

$$cr_D(\Gamma(Z_{3p})) = \sum_{i=1}^{2} \sum_{j=1}^{p-1} cr_D(v_i, u_j)$$

*The proof was based on having all of the vertices on the X and Y axis. The two vertices in* $V_1$ *are placed in X axis and the p-1 vertices in* $V_2$ *are placed in Y axis. Then, we connect all the vertices on the X axis with all the vertices on the Y axis. Clearly,* $\Gamma(Z_{3p})$ *is a planar graph. and* $cr(V_1) = 0$ *and* $cr_{V_2} = 0$, *or* $cr(u_1) = cr(u_2) = ...... = cr(u_{(p-1)}) = 0$. *Hence,* $cr(\Gamma(Z_{3p}))=0$.

**Theorem 2.4** *For any graph* $\Gamma(Z_{5p})$, *where* $p > 5$ *is any prime, then* $cr(\Gamma(Z_{5p})) = 2\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor$.

***Proof:*** *The vertex set of* $\Gamma(Z_{5p})$ *is* $\{5, 10, ...., 5(p-1), p, 2p, 3p, 4p\}$. *Clearly* $|V(\Gamma(Z_{5p}))| = p+3$. *Let u and v be any two vertices in* $\Gamma(Z_{5p})$ *with maximum and minimum degree, respectively. Let* $u = p$ *and* $v = 10$ *then* $5p$ *must divide* $uv$ *which implies that u and v are adjacent.*

*Let* $u = p$ *and* $v = 2p$ *then* $5p$ *does not divide* $uv = 2p^2$, *which implies that u and v are non-adjacent vertices. Then the vertex set V can be partitioned into two parts* $V_1$ *and* $V_2$, *where* $V_1 = \{p, 2p, 3p, 4p\}$ *and* $V_2 = \{5, 10, ...., 5(p-1)\}$. *Clearly any vertices u and v in* $V_1$ *are non-adjacent as same as in* $V_2$. *let* $u = p$ *in* $V_1$ *and* $v = 10$ *are in* $V_2$ *then* $5p$ *divides* $uv = 10p$. *Finally we note that, every vertex in* $V_1$ *is adjacent to all the vertices in* $V_2$. *Moreover* $V(\Gamma(Z_{5p})) = V_1 \cup V_2$ *and* $V_1 \cap V_2 = \phi$.

*Let, D be a good drawing of* $\Gamma(Z_{5p})$. *Next we check that D is a drawing that satisfies the following conditions.*

*i) No edges crosses itself, ii) Adjacent edges do not cross each other and iii) Non adjacent edges cross each other atmost once.*

*The proof is by the method of induction. If* $p = 7 > 5$, *then* $|V_1| = 4$ *and* $|V_2| = p - 1 = 6$. *Let the elements of* $V_1$ *be* $v_1, v_2, v_3, v_4$ *and the elements of* $V_2$ *be* $u_1, u_2, ....u_{p-1}$. *In a drawing D, we denote by* $cr_D(v_i, u_j)$ *the number of crossings of arcs, one terminating at* $v_i$, *the other at* $u_j$ *and by* $cr_D(v_i)$ *the number of crossings arcs which terminate at* $v_i$,

$$cr_D(v_i) = \sum_{j=1}^{6} cr_D(v_i, u_j)$$

*Clearly, the crossing number of* $\Gamma(Z_{5p})$ *is*

$$cr_D(\Gamma(Z_{5p})) = \sum_{i=1}^{4} \sum_{j=1}^{6} cr_D(v_i, u_j)$$

*Since, the crossing number of* $\Gamma(Z_{5p})$ *is either the sum of the crossing number of the vertices in* $V_1$ *or the sum of the crossing number of the vertices in*

*$V_2$. Now, we consider the vertex set $V_1$. The proof was based on having all of the vertices on the X and Y axis. The first thing is to place $\left\lfloor \frac{p-1}{2} \right\rfloor$ vertices on one side of the X axis and $\left\lfloor \frac{p}{2} \right\rfloor$ vertices on the other side. Then we connect all the vertices on the X axis with all the vertices on the Y axis.*

*By the hypothesis, the crossing number of a vertex $v_1$ is denoted by $cr(v_1)$. Finally, we compute the crossing number of $\Gamma(Z_{5p})$. We get $cr(v_1) = 0$, $cr(v_2) = 0$, $cr(v_3) = 6$, $cr(v_4) = 6$.*

$$\begin{aligned} cr(\Gamma(Z_{5p})) &= \textstyle\sum_{i=1}^{4} cr(v_i) = 12 \\ &= 2 \times 3 \times 2 \\ &= 2 \left\lfloor \frac{7-1}{2} \right\rfloor \left\lfloor \frac{7-2}{2} \right\rfloor \\ &= 2 \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor. \end{aligned}$$

**Theorem 2.5** *For any graph $\Gamma(Z_{7p})$, where $p$ is any prime $> 7$, then $cr(\Gamma(Z_{7p})) = 6 \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor$.*

***Proof:*** *The vertex set of $\Gamma(Z_{7p})$ is $\{7, 14, ..., 7(p-1), p, 2p, 3p, 4p, 5p, 6p\}$. Let $u = 7$ and $v = p$ in $V(\Gamma(Z_{7p}))$. Then $7p$ must divide $uv$, which implies that $u$ and $v$ are adjacent vertices in $\Gamma(Z_{7p})$. Let $u = 7$ and $v = 14$ then $7p$ does not divide $uv$. It seems that, the vertex set of $\Gamma(Z_{7p})$ can be partitioned into two parts $V_1$ and $V_2$. Clearly no two vertices in $V_1$ are adjacent same as in $V_2$. Next, we calculate either the sum of the crossing number of the vertices in $V_1$ or the sum of the crossing number of the vertices in $V_2$. By the method of induction, let us assume that $p = 11$.*

*Using theorems (2.3) and (2.4), the six vertices $\{p, 2p, 3p, 4p, 5p, 6p\}$ in $V_1$ are placed in X axis and $(p-1)$ vertices $\{7, 14, ..., 7(p-1)\}$ in $V_2$ are placed in Y axis. Then, we connect all the vertices on the X axis with all the vertices in the Y axis. Clearly, $cr(p) = 0$, $cr(2p) = 0$, $cr(3p) = 20$, $cr(4p) = 20$, $cr(5p) = 40$ and $cr(6p) = 40$. Then,*

$$cr(\Gamma(Z_{7p})) = \textstyle\sum_{i=1}^{6} \sum_{j=1}^{p-1} cr_D(v_i, u_j), \text{ where, } v_i \in V_1 \text{ and } u_j \in V_2.$$

$$\begin{aligned} cr(\Gamma(Z_{7p})) &= \text{Sum of the crossing number in } V_1 \text{ or in } V_2 \\ &= cr(p) + cr(2p) + cr(3p) + cr(4p) + cr(5p) + cr(6p) \\ &= 0 + 0 + 20 + 20 + 40 + 40 = 120 = 6 \times 5 \times 4 \end{aligned}$$

$$Hence, cr(\Gamma(Z_{7p})) = 6 \left\lfloor \frac{11-1}{2} \right\rfloor \left\lfloor \frac{11-2}{2} \right\rfloor = 6 \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor.$$

**Theorem 2.6** *If $p$ and $q$ are distinct prime numbers with $q > p$ then, $cr(\Gamma(Z_{pq})) = \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor \left\lfloor \frac{q-2}{2} \right\rfloor$.*

***Proof:*** *The vertex set of $\Gamma(Z_{pq})$ is $\{p, 2p, 3p, ..., p(q-1), q, 2q, 3q, ..., (p-1)q\}$. Let $v = p$ and $w = p(q-1)$ in $\Gamma(Z_{pq})$. Then $pq$ does not divides $uw = p^2(q-1)$. Clearly $v$ and $w$ are non-adjacent vertices. Let $u = q$ and $v = p$ then $pq$ must divides $uv$, which implies that $u$ and $v$ are adjacent vertices. So the vertex set $V$ can be partitioned into two parts $V_1$ and $V_2$ which implies that the vertex $p$, multiples of $p$ are in $V_1$ and $q$, multiples of $q$ are in*

$V_2$. *Clearly every two vertices in* $V_1$ *are non-adjacent same as in* $V_2$. *Then,*
$$|V| = |V_1| + |V_2| = p - 1 + q - 1 = p + q - 2.$$

*Using theorem* (2.5), *the vertices in* $V_1$ *are placed in* $X$ *axis and the vertices in* $V_2$ *are placed in* $Y$ *axis. Let* $D$ *be a good drawing of* $\Gamma(Z_{pq})$, *then* $D$ *satisfies, no edges crosses itself, adjacent edge do not cross each other and non adjacent edges cross each other atmost one. Then, we connect all the vertices on the* $X$ *axis with all the vertices on the* $Y$ *axis. The proof is by the method of induction on the number of vertices in* $\Gamma(Z_{pq})$.

***case(i):*** *Let* $p = 2$, $q > p$.

*Using theorem* (2.2), $cr(\Gamma(Z_{pq})) = 0$.

***case(ii):*** *Let* $p = 3$, $q > p$.

*Using theorem* (2.3), $cr(\Gamma(Z_{pq})) = 0$.

***case(iii:*** *Let* $p = 5$, $q > p$.

*Using theorem* (2.4), $cr(\Gamma(Z_{7q})) = 6 \left\lfloor \frac{q-1}{2} \right\rfloor \left\lfloor \frac{q-2}{2} \right\rfloor = 3 \times 2 \times \left\lfloor \frac{q-1}{2} \right\rfloor \left\lfloor \frac{q-2}{2} \right\rfloor$
$$= \left\lfloor \frac{5-1}{2} \right\rfloor \left\lfloor \frac{5-2}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor \left\lfloor \frac{q-2}{2} \right\rfloor$$
$$= \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor \left\lfloor \frac{q-2}{2} \right\rfloor.$$

***case (iv):*** *Let* $p = 7$, $q > p$.

*Using theorem* (2.5), $cr(\Gamma(Z_{7q})) = 6 \left\lfloor \frac{q-1}{2} \right\rfloor \left\lfloor \frac{q-2}{2} \right\rfloor = 3 \times 2 \times \left\lfloor \frac{q-1}{2} \right\rfloor \left\lfloor \frac{q-2}{2} \right\rfloor$
$$= \left\lfloor \frac{7-1}{2} \right\rfloor \left\lfloor \frac{7-2}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor \left\lfloor \frac{q-2}{2} \right\rfloor$$
$$= \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor \left\lfloor \frac{q-2}{2} \right\rfloor.$$

*Continuing the above process,* $cr(\Gamma(Z_{pq})) = \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor \left\lfloor \frac{q-2}{2} \right\rfloor.$

**Theorem 2.7** *If* $p$ *is any prime, then* $cr(\Gamma(Z_{p^2})) = \frac{1}{4} \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor \left\lfloor \frac{p-3}{2} \right\rfloor \left\lfloor \frac{p-4}{2} \right\rfloor.$

***Proof:*** *The vertex set of* $\Gamma(Z_{p^2})$ *is* $\{p, 2p, 3p, ..., (p-1)p\}$. *Clearly,* $p$ *is adjacent to all the vertices in* $V(\Gamma(Z_{p^2}))$. *Also note that any two vertices in* $\Gamma(Z_{p^2})$ *is adjacent and* $|V(\Gamma(Z_{p^2}))| = p - 1$. *Clearly,* $\Gamma(Z_{p^2})$ *is a* $k_{p-1}$ *graph* [6].

*Since,* $p$ *is any prime number,* $(p-1)$ *is even. place,* $(p-1)/2$ *of the vertices on the inner circle and* $(p-1)/2$ *of the vertices on the outer circle. Connect the vertices on the inner circle with those on the outer circle by drawing* $(p-1)(p-2)/2$ *edges. The proof is by the induction method on the vertices in* $V(\Gamma(Z_{p^2}))$.

***case(i):*** *Let* $p = 2$ *or* $p = 3$.

*Using theorems* (2.2) *and* (2.3), $cr(\Gamma(Z_{p^2})) = 0$.

***case (ii):*** *Let* $p = 5$.

*The vertex set of* $\Gamma(Z_{p^2})$ *is* $\{5, 10, 15, 20\}$. *Clearly,* $\Gamma(Z_{p^2})$ *is a* $K_4$ *graph and* $\Gamma(Z_{p^2})$ *is planar graph. Hence,* $cr(\Gamma(Z_{p^2})) = 0$.

***case (iii):*** *Let* $p = 7$.

*The vertex set of* $\Gamma(Z_{p^2})$ *is* $\{7, 14, 21, 28, 35, 42\}$. *Clearly, place any three vertices on the inner circle and place the remaining three vertices on the outer* *circle. Connect the vertices on the inner circle with those on the outer circle by drawing* 15 *edges. Let, the vertex set of* $\Gamma(Z_{p^2})$ *be* $\{v_1, v_2, v_3, v_4, v_5, v_6\}$*. It seems that,* $c(v_1) = 0, c(v_2) = 0, c(v_3) = 0, c(v_4) = 0, c(v_5) = 1, c(v_6) = 2.$

$$cr(\Gamma(Z_{p^2})) = \sum_{i=1}^{6} cr_D(v_i)$$
$$= 3 = 1/4 \times 3 \times 2 \times 2 \times 1$$
$$= \frac{1}{4}\left\lfloor\frac{7-1}{2}\right\rfloor\left\lfloor\frac{7-2}{2}\right\rfloor\left\lfloor\frac{7-3}{2}\right\rfloor\left\lfloor\frac{7-4}{2}\right\rfloor.$$
$$= \frac{1}{4}\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor\left\lfloor\frac{p-3}{2}\right\rfloor\left\lfloor\frac{p-4}{2}\right\rfloor.$$

*Continuing the above process, assigning any prime value for p, we get the crossing number of* $\Gamma(Z_{p^2})$ *as* $\frac{1}{4}\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor\left\lfloor\frac{p-3}{2}\right\rfloor\left\lfloor\frac{p-4}{2}\right\rfloor$.

**Theorem 2.8** *For any prime* $p \geq 3$*, then* $cr(\Gamma(Z_{4p})) = (p-1)(p-3)/4$.

***Proof:*** *The vertex set of* $\Gamma(Z_{4p})$ *is* $\{2, 4, 6, 8, 10, 12, ..., 2(2p-1), p, 2p, 3p\}$ *and* $|V(\Gamma(Z_{4p}))| = 2p+1$*. Let,* $u = p$ *and* $w = 3p$ *then* $uw = p.3p = 3p^2$ *is not divisible by* $4p$*. Clearly,* $u$ *and* $w$ *are non-adjacent vertices.*

*Let* $x \in \{4, 8, 12, ...., 4(p-1)\}$ *then* $x$ *is adjacent to both* $u$ *and* $w$*. That is,* $4p$ *must divide* $xu$ *and* $xw$*. Similarly, let* $y \in \{2, 4, 6, ..., 2(2p-1)\}$ *then* $y$ *is adjacent to v=2p, which implies that 4p divides yv. Clearly, the vertex set of* $V(\Gamma(Z_{4p}))$ *can be partitioned into two parts,* $V_1 = \{p, 2p, 3p\}$*,* $V_2 = \{2, 4, 6, 8, ...., 2(2p-1)\}$ *which implies that no two vertices in* $V_1$ *are non adjacent same as in* $V_2$*. Clearly,* $\Gamma(Z_{4p})$ *is a bipartite graph, which is not complete.*

*The vertices in* $V_1$ *are placed in* $X$ *axis and the vertices in* $V_2$ *are placed in* $Y$ *axis. Then, we connect all the vertices on the* $X$ *axis with all the vertices on the* $Y$ *axis. Using the method of induction on p finally we get,* $cr(\Gamma(Z_{4p})) = (p-1)(p-3)/4$.

**Note:**
(i) If $p = 5$, then $cr(\Gamma(Z_{4p})) = 2$.
(ii) If $p = 7$, then $cr(\Gamma(Z_{4p})) = 6$.
(iii) If $p = 11$, then $cr(\Gamma(Z_{4p})) = 20$, and so on.

**Theorem 2.9** *For any graph* $\Gamma(Z_{pqr})$ *with* $p = 2$*,* $q = 3$ *and* $r$ *is any prime then,* $cr(\Gamma(Z_{pqr})) = (r-1)(2r-5)/2$.

**Theorem 2.10** *For any graph* $p \geq 5$*, then* $cr(\Gamma(Z_{9p})) = 12\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor$.

# 3 New Conjecture of Complete and Complete Bipartite Graphs in Zero Divisor Graph

In this section, we posted two conjecture of complete and complete bipartite graph in zero divisor graph. We know that, determining the crossing number of the complete bipartite graph is one of the oldest crossing number open problems. In 1954, Zarankiewicz[9] conjectured that it is equal to

$$Z(m,n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor .$$

Using theorem (2.6), $\Gamma(Z_{pq})$ is a complete bipartite graph, namely $K_{p-1,q-1}$. Now, we define the new bound of $K_{p-1,q-1}$ as

$$\mathbf{cr(\Gamma(Z_{pq})) = cr(K_{p-1,q-1}) = \frac{(p-1)(p-3)(q-1)(q-3)}{16}}.$$

In 1960, Guy [3] popularized the search for the crossing number of a complete graph with the introduction of an upper bound, confirmed by Blazek and Koman. The crossing number of the complete graph $K_n$ satisfies the inequality,

$$cr(K_n) \leq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor .$$

Using theorem (2.7), $\Gamma(Z_{p^2})$ is a complete graph, namely $K_{p-1}$. Now, we define the new bound of $K_{p-1}$ as

$$\mathbf{cr(\Gamma(Z_{p^2})) = cr(K_{p-1}) = \frac{(p-1)(p-3)^2(p-5)}{64}}.$$

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