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Semigroups of Quasi-Open Mappings and Lattice-Equivalence

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Abstract

In this paper we consider the semigroups of quasi-open functions. A mapping f between topological spaces X and Y is quasi-open if for any non-empty open set $U \subset X$, the interior of $f(U)$ in Y is non-empty. We give an abstract characterization of semigroups of quasi-open mappings defined on a certain class of topological spaces.

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1 Introduction

In [5] Thron introduced a concept of lattice-equivalence of topological spaces. Let X be a topological space and let $C(X)$ be the lattice of closed sets of X. Two topological spaces X and Y are said to be lattice equivalent if there is a bijective map from $C(X)$ onto $C(Y)$ which together with its inverse is order preserving. Thron proved among others that for T_D-spaces X and Y , any lattice-isomorphism $\phi : C(Y) \to C(X)$ can be induced by a homemorphism $f : X \to Y$. It is worth noting that several researchers dealt with the concept of lattice equivalent topological spaces and representations of an abstract lattice as the family of closed sets on a topological space [1], [6]. Several researchers focused their efforts on the characterization of topological spaces by semigroups of continuous, open, and closed mappings [4], [7]. A map f between topological spaces X and Y is quasi-open if for any non-empty open set $U \subset X$, the interior of $f(U)$ in Y is non-empty. The quasi-open maps have the properties similar to those of the continuous maps. But the quasi-open maps and the continuous maps are not related. Some characterizations of M_1-spaces, in terms of quasi-open maps given by Kao in [2]. If f and g are both quasi-open, then the function composition is also quasi-open. Let $Q\,(X)$ denote the semigroup of quasi-open maps from a topological space X into itself with composition of functions as multiplication. The purpose of this paper is to investigate semigroups of quasi-open maps in light of lattice-equivalence. It is obvious that if X and Y are homeomorphic then the semigroups $Q\,(X)$ and $Q\,(Y)$ are isomorphic. If $Q\,(X)$ and $Q\,(Y)$ are isomorphic, must X and Y be homeomorphic? In general, the answer is no. Let X denote any set with more than two elements containing the elements η, ξ. Consider the topological spaces $Y = (X, \tau_1)$ and $Z = (X, \tau_2)$ with $\tau_1 = \{\oslash, \{\eta\}, X\}$ and $\tau_2 = \{\oslash, \{\eta\}, X \setminus \{\xi\}, X\}$. Evidently $Q\,(Y)$ and $Q\,(Z)$ are isomorphic but Y and Z are not homeomorphic. In this paper, we give an abstract characterization of semigroups of quasi-open maps for a certain class of topological spaces.

2 An Abstract Characterization of Semigroups of Quasi-Open Maps

A topological space X is said to be a T_D-space if for every point ξ in X the set $\overline{\{\xi\}} \setminus \{\xi\}$ is closed [5]. We denote the set $\overline{\{\xi\}} \setminus \{\xi\}$ by $\{\xi\}'$. Obviously, each T_D-space is T_0-space and each T_1-space is T_D-space. We call a topological space X a T_D^+-space if it is a T_D-space with no one-point open sets and if for every point ξ in X and for every open set U containing ξ the set $U \cap \left(X \setminus \overline{\{\xi\}}\right)$ is not empty. Note that each T_1-space without isolated points is T_D^+-space.

Lemma 1 *Let X be a T_D^+-space and let $\xi \in X$ and let a, b be arbitrary elements of $Q\,(X)$. The condition*

$$\forall f, g \in Q(X), fa = ga \rightarrow fb = gb \tag{1}$$

is necessary and sufficient for $b(X) \subseteq a(X)$.

Proof. If the condition $b(X) \subseteq a(X)$ is satisfied, then for every $x \in X$ there exists a point $\xi \in X$ such that $b(x) = a(\xi)$. Then

$$fb(x) = f(b(x)) = f(a(\xi)) = fa(\xi) = ga(\xi) = g(a(\xi)) = g(b(x)) = gb(x).$$

So, condition (1) holds.

Now let condition (1) hold for some $a, b \in Q(X)$. Suppose that the set $b(X) \setminus a(X)$ is not empty. For any point $\xi = b(x)$ in $b(X) \setminus a(X)$ there exist $f, g \in Q(x)$, such that $f\,(\xi) \neq g\,(\xi)$ but $f\,(x) = g\,(x)$ for all $x \in X \setminus \{\xi\}$. Indeed, select a point ξ in X and consider the map $f : X \rightarrow X$ defined by

$$f(x) = \begin{cases} \eta_1 & \text{if } \ x = \xi \\ x & \text{if } \ x \neq \xi \end{cases}$$

and the map $g : X \to X$ defined by

$$g(x) = \begin{cases} \eta_2 & \text{if } x = \xi \\ x & \text{if } x \neq \xi \end{cases}$$

where $\eta_1 \neq \eta_2$ are any fixed points in $X \setminus \{\xi\}$. The maps f and g are quasi-open and we have $f(\xi) \neq g(\xi)$ but $f(x) = g(x)$ for all $x \in X \setminus \{\xi\}$. Then for every $x \in X$ the point $a(x)$ is in $X \setminus \{\xi\}$ and therefore $fa(x) = f(a(x)) = g(a(x)) = ga(x)$. But for $\xi = b(x) \in b(X) \setminus a(X)$ we have $fb(x) = f(b(x)) = f(\xi) \neq g(\xi) = g(b(x)) = gb(x)$ which contradicts to (1). ∎

Lemma 2 *Let X and Y be T_D^+-spaces and let $\varphi : Q(X) \to Q(Y)$ be an isomorphism between semigroups $Q(X)$ and $Q(Y)$. If $a(X) \subseteq b(X)$ for some $a, b \in Q(X)$ then $(\varphi a)(Y) \subseteq (\varphi b)(Y)$. Hence if $a(X) = b(X)$ for some $a, b \in Q(X)$ then $(\varphi a)(Y) = (\varphi b)(Y)$.*

Proof. Suppose that $b(X) \subseteq a(X)$. If $f'(\varphi a) = g'(\varphi a)$ for some elements $f', g' \in Q(Y)$ then there exist $f, g \in Q(X)$ such that $f' = \varphi f$ and $g' = \varphi g$. Then $(\varphi f)(\varphi a) = (\varphi g)(\varphi a)$ and since φ is an isomorphism, $\varphi(fa) = \varphi(ga)$ and $fa = ga$. We have $fb = gb$, by Lemma 1. Again, since φ is an isomorphism, then $(\varphi f)(\varphi b) = (\varphi g)(\varphi b)$ and therefore $f'(\varphi b) = g'(\varphi b)$. Because $f'(\varphi b) = g'(\varphi b)$ is true for every $f', g' \in Q(Y)$ satisfying the condition $f'(\varphi a) = g'(\varphi a)$ it follows from Lemma 1 that $(\varphi b)(Y) \subseteq (\varphi a)(Y)$. In the same way, we could show that if $a(X) \subseteq b(X)$ then $(\varphi a)(Y) \subseteq (\varphi b)(Y)$. ∎

Let X be a T_D^+-space that has an open base, each element of which is an image of X under a quasi-open mapping and let Λ be a class of all such spaces. For instance, the open subsets of the α-cube I^α, $\alpha \geqslant 1$, the set $\mathbb{R}$ of real numbers with Zariski topology and any topological space X, $|X| \geqslant \aleph_0$, with cofinite topology belong to the class Λ.

Lemma 3 *Let $X \in \Lambda$ and let U be any open subset of X. Then there exists a quasi-open mapping $a \in Q(X)$ such that $a(X) = U$.*

Proof. Let $X \in \Lambda$ and $\Im$ is an open base of X. Suppose that U is an open subset of X and $i : U \to X$ is the inclusion map, which is open map. Let $V_1 \in \Im$ and $V_1 \subset U$, then there exists a quasi-open mapping f from X onto V_1. Consider the restriction of f to $X \setminus \overline{U}$. Since restriction of a quasi-open map to an open set is quasi-open, this map is quasi-open. Denote by g the extension of this mapping to $X \setminus U$ obtained by assigning all boundary points of U to any fixed point in U. The mapping $a : X \to U$ defined by

$$a(x) = \begin{cases} i(x), & \text{if } x \in U \\ g(x), & \text{if } x \in X \setminus U \end{cases}$$

is a quasi-open map and $a(X) = U$. ■

Let X be a topological space. The family $O(X)$ of all open sets of X is a complete distributive lattice if set inclusion is taken as the ordering. By the duality principle for ordered sets, two topological spaces X and Y are homeomorphic if and only if lattices $O(X)$ and $O(Y)$ are isomorphic [5].

Theorem 4 *Let $X, Y \in \Lambda$. If the semigroups $Q(X)$ and $Q(Y)$ are isomorphic then the lattices $O(X)$ and $O(Y)$ are lattice-isomorphic.*

Proof. Let U be any open subset of X. By Lemma 3 there exists a quasi-open function $a \in Q(X)$ such that $a(X) = U$. Since the semigroups $Q(X)$ and $Q(Y)$ are isomorphic there exists a quasi-open function $a' \in Q(Y)$ such that $\varphi a = a'$. Let $a'(Y) = U'$. We define a map θ from $O(X)$ to $O(Y)$ by assinging to each open set $U \subset X$ the set $U' \subset Y$. The map θ does not depend on the choice of $a \in Q(X)$. Indeed , if $a(X) = U$ and $b(X) = V$ then Lemma 2 says that $(\varphi a)(Y) = (\varphi b)(Y) = U'$. Let U and V be any two different open subsets of X. By Lemma 3 there exist two quasi-open functions $a, b \in Q(X)$ such that $a(X) = U$ and $b(X) = U$. Since the semigroups $Q(X)$ and $Q(Y)$ are isomorphic it follows from Lemma 2 that $(\varphi a)Y \neq (\varphi b)Y$. Hence θ is bijective. Now suppose that U' is an arbitrary open set in Y. Since the semigroups $Q(X)$ and $Q(Y)$ are isomorphic it follows from Lemma 2 that there exists an open set $U \subset X$ such that $\theta(U) = U'$. Again it follows from Lemma 2 that if $U \subseteq V$ then $\theta(U) \subseteq \theta(V)$. From Theorem 2.1 of [5] it follows that the topological spaces X and Y are homeomorphic. ■

Theorem 5 *Let $X, Y \in \Lambda$. The semigroups $Q(X)$ and $Q(Y)$ are isomorphic if and only if the spaces X and Y are homeomorphic.*

Proof. It is obvious that if X and Y are homeomorphic then $Q(X)$ and $Q(Y)$ are isomorphic. Specifically, if h is a homeomorphism from X onto Y, then $f \to h \circ f \circ h^{-1}$ is an isomorphism from $Q(X)$ onto $Q(Y)$. The proof of the necessary condition follows from Theorem 4. ■

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