International Journal of Algebra, Vol. 6, 2012, no. 31, 1483 - 1490

$E_\infty-$Coalgebra with Filtration and Chain Complex of Simplicial Set

A. Nasser

Department of Mathematics, Faculty of Science
Sanaa University, Yemen
manasser68@yahoo.com

Y. Gh. Gouda

Department of Mathematics
Aswan Faculty of Science Aswan University, Egypt
yasiengouda@yahoo.com

Abstract

The paper deals with differntial $E_\infty-$(co)algebra with filtration. It is shown that the chain (cochain) complex of simplicial set is differential $E_\infty-$coalgebra (differential $E_\infty-$algebra) with filtration.

Mathematics Subject Classification: 55U35

Keywords: $E_\infty-$Coalgebra- operad- filtration- simplicial set

1 Introuduction

It is well known that the great part of classical theory of spectral sequence depends on the concept of differential module with filtratin.In [1] Leray initied and studied this conception to construct the space of spectral sequence. Serre in [2] had shown that the cochain (chain) complex of simplicial set is a differential algebra (co algebra) with filtration. In [6] Smirnov introduced the concept of $E_\infty-$algebra in category of differetial modules, one of the main properties of the structure of an $E_\infty-$algebra is its homotopy invariance; that is, the structure of an $E_\infty-$algebra has the following property: in each differential graded module homotopy equivalent to an E_∞-algebra one can also introduce the structure of an $E_\infty-$algebra.

In the present work, is given the differntial $E_\infty-$(co)algebra with filtration. We show that the chain (cochain) complex of simplicial set is differential $E_\infty-$coalgebra (differential $E_\infty-$algebra) with filtration.

Firstly we recall some definitions and facts related to differential modules, filtration, operad and algebra over operad useful in the sequal.The main references are [3],[7],[4] and [5].

Definition 1 *The differential graded module* $(X.d)$ *is any graded module* $X = \{X_n\}, n \in Z$ *, equipped with a differential* $.d = M_\bullet \to M_{\bullet-1}$ *such that* $d^2 = 0$.

Definition 2 *A differential modules map* $f : (X, d) \to (Y, d)$ *is a graded modules* $f : X_\bullet \to Y_\bullet$ *with degree zero, such that* $df = fd$*.The* hom*otopy* $h : X \to Y$ *between the maps* $f, g : (X, d) \to (Y, d)$ *is a graded module homomorphism*

$h : X_\bullet \to Y_{\bullet+1}$,such that $dh + hd = f - g$.

Definition 3: A filtration $\{X^n\}$ of differential module (X, d), is a family of graded submodules $X^n_\bullet \subseteq X_\bullet$, $n \in Z$ satisfying the following conditions:

$... \subseteq X^n_\bullet \subseteq X^{n+1}_\bullet \subseteq ...,\ \bigcup_{n\in Z} X^n = X,\ \bigcap_{n\in Z} X^n = 0,\ d(X^n) \subseteq X^n,\ n \in Z.$

Definition 4: A map $f : (X, \{X^n\}) \to (Y, \{Y^n\})$ of differential modules with filtrations is a map $f : X \to Y$ of differential modules satisfying the condition $f(X^n) \subseteq Y^n$, $n \in Z$.

Definition 5: A homotopy h between maps of differential modules $f, g : (X, \{X^n\}) \to (Y, \{Y^n\})$ with

filtrations is by definition a homotopy $h : X \to Y$ between the maps of differential modules $f, g : X \to Y$ satisfying the condition $h(X^n) \subseteq Y^n$, $n \in Z$.

Definition 6 : A (1)-filtration of a differential module (X, d) is an arbitrary filtration $\{X^n\}$ of this differential module satisfying the condition $d(X^n) \subseteq X^{n-1}$, $n \in Z$., Maps of differential modules with (1)-filtrations are defined as maps of differential modules with filtrations. A homotopy h between maps of differential modules

with (1)-filtrations $f, g : (X, \{X^n\}) \to (Y, \{Y^n\})$ is any homotopy $h : X \to Y$ between the maps of differential modules $f, g : X \to Y$ satisfying the condition $h(X^n) \subseteq Y^{n+1}$, $n \in Z$.

Note that : a) The category of differential modules with (1)-filtrations is a full subcategory of the category of differential modules with filtrations.

b) The embedding functor from the category of differential modules with (1)-filtrations into the category of differential modules with filtrations does not preserve homotopies between morphisms and, consequently, does not induce any functor.

2 Operad and algebra over operad

Definition 7 : The symmetry family $O\{O(j)\}_{j\geq 1}$, in the category of differential modules is a collection of differential modules $O(j)$,Carrying the right action of symmetric group $\sum_j$, $j \geq 1$.

A morphism of symmetric families $f : O' \to O''$ is defined as a family of $\sum_j$ −equivariant maps of differential modules $f : \{f(j) : O'(j) \to O''(j)\}_{j\geq 1}$.

Definition8 : . An operad (O, π) is a symmetric family $O\{O(j)\}_{j\geq 1}$ considered together with a morphism of symmetric families $\pi : O \times O \to O$, such that $\pi(\pi\times 1) = \pi(1\times\pi)$. Clearly the collection $O\times O = \{(O\times O)(j)\}_{j\geq 1}$is a symmetric family such that the differential $\sum_j$ − module $(O \times O)(j)$ is defined as a differential factor module

$$\underset{j_1+...+j_k=j}{\oplus}(O(k) \otimes (O(j_1) \otimes ... \otimes O(j_k) \otimes \sum_j) \,/\,, k \geq 1,$$

since the equivalent relation satisfied the following relations

$$e\sigma \otimes e_1 \otimes ... \otimes e_k \;\; (e \otimes e_{\sigma^{-1}(1)} \otimes ... \otimes e_{\sigma^{-1}(k)})\sigma(j_1, ..., j_k),$$
$$e \otimes e_1\sigma_1 \otimes ... \otimes e_k\sigma_k \;\; (e \otimes e_1 \otimes ... \otimes e_k)(\sigma_1 \times ... \times \sigma_k),$$

where $\sigma(j_1, ..., j_k) \in \sum_j$, $(\sigma_1\times ...\times\sigma_k)$ is the image of element $\sigma_1\times ...\times\sigma_k \in \sum_{j_1} \times ... \times \sum_{j_k} \in \sum_j$ A morphism of operads $f : (O', \pi') \to (O'', \pi'')$ is defined as a morphism of symmetric families $f : O' \to O''$ satisfying the condition $f\pi' = \pi''(f \times f)$.

A canonical example of an operad[4] is (O_X, π), which can be defined for each graded differential module X. Recall the construction of this operad. Let $X^{\otimes j}$ be the j times tensor product K of the graded differential module X. We consider the graded differential module $O_X(j)$ defined by setting $O_X(j) = Hom_K(X^{\otimes j}, X)$, where $Hom_K(X^{\otimes j}, X)$ is the graded differentiall module of all homomorphisms of the form $X^{\otimes j}$ to X. The action of the symmetric group$\sum_{j_1}$ on $O_X(j)$ is defined by the action of $\sum_{j_1}$ on $X^{\otimes j}$ by rearrangements of the factors. The operad structure on the symmetric family O_X , that is, multiplication $\pi : O_X\times O_X \to O_X$, is defined by the following formula:

$\pi(g\otimes g_1\otimes\cdots\otimes g_k)=g\circ(g_1\otimes\cdots\otimes g_k), g_i\in O_X(j_i),\ 1\leq i\leq k,\ g\in O_X(k).$

Note that we can define in similar fashion the operad (O^X,π) whose structure is given by the following equalities:

$O^X(j)=Hom_K(X,X^{\otimes j}),\ \pi(g\otimes g_1\otimes\cdots\otimes g_k)=(g_1\otimes\cdots\otimes g_k)\circ g.$

A one of an important operad in algebraic topologyis defined by simirnov [6] is an operad (E_∞,π),where$E_\infty=\{E_\infty(j)\}_{j\geq1}$ such that E_∞ is free,acyclic and .$\sum$-free operad., for example, the differential module $E_\infty(2)$ is a $K[Z_2]$-free acyclic chain complex with generators $\cup_i\in E_\infty(2)$ of dimension $i\geq 0$ and with the boundary operator $d(\cup_i)=\cup_{i-1}+(-1)^i\cup_{i-1}T,\ T\in\sum_2=Z_2$.

Definition 9 : Algebra (X,μ) over operad (O,π),in short, $O-$algebra is differential module X together with the morphism operad $\mu: O\to O_X$. Similarly we can define the coalgebra ($O-$coalgebra) over operad (O,π), by differential module X together with the morphism operad $\upsilon: O\to O^X$. Clearly that the structure of morphism operad μ of arbitrary $O-$algebra is given by the following differential module morphism $\mu=\{\mu_j: O(j)\otimes_{\sum_j}X^{\otimes j}\to X\}_{j\geq1}$with the following condition

$$\mu_j(\pi(e\otimes e_1\otimes\cdots\otimes e_k)\otimes(x_1\otimes\cdots\otimes x_j)=$$

$$=\mu_j(e\otimes\mu_{j_1}\otimes(e_1\otimes x_1\otimes\cdots\otimes x_{j_1})\otimes...\otimes\mu_{j_k}(e_k\otimes x_{j_1+...+\ j_{k-1}+1}\otimes\cdots\otimes x)),$$

where $e\in O(k),\ e_i\in O(j_i),\ 1\leq i\leq k,\ j=j_1+...+j_k$.

Note that , If a differential module is $O-$algebra (X,μ) , then the dual differential X^* module has is $O-$coalgebra (X^*,υ), where such that for $e\in O(j),\xi\in X^*,\ x_i\in X, 1\leq i\leq j$,we have :

$$\upsilon(e\otimes\xi))(x_1\otimes\cdots\otimes x_j)=\xi(\mu(e\otimes x_1\otimes\cdots\otimes x_j)).$$

Similarly If a differential module is $O-$coalgebra (X,υ) , then the dual differential X^* module has is $O-$algebra , where such that for $e\in O(j),\xi_i\in X^*, 1\leq i\leq j,\ x\in X$,we have :

$$(\mu(e\otimes\xi_1\otimes\cdots\otimes\xi_j))(x)=\xi_1\otimes...\otimes\xi_j)(\upsilon(e\otimes x)).$$

Recall from [2] that is a graded differential set $X=\{X_n\}, n\in z,\ n\geq 0$, elements in X_n is called n-degree simplex of this simplicial set. Above all for every $n\geq 0$ there are two familes of operators act on simplicial set $X=\{X_n\}$:

$$\partial_i : X_n \to X_{n-1},\ s_i : X_n \to X_{n+1},\ 0 \le i \le n.$$

Consioder the simplicial subset X^n in X with an inclusion $X^n \subseteq X^{n+1}$. The family $\{X^n\}_{n\ge 0}$ of simplicial set X is called a filtration of simplicial set X. Clearly for any simplicial set X we can define a pair $(C(X), \partial)$,where $C(X) = \{C_n(X)\},\ n \ge 0$ which is called a chain complex of simplicial set X and the differential $\partial_i : C_n(X) \to C_{n-1}(X)$ is given by : $\partial(x) = \partial_0(x) - \partial_1(x) + ... + (-1)^n\partial_n(x),\ x \in X_n$.

The dual $C^*(X)$ of a differential module $C(X)$ is called a cochain complex of simplicial set X.

An important vairiant structure of chain complex $C(X)$of differential module X is normalized chain complex $N(X)$,which is a factor -complex of chain complex $C(X)$ by subcomplex generated by simplex of simplicial set X . The dual of $N(X)$ is denoted by $N^*(X)$ and called normalized cochain complex of X .Clearly the copmplexes $N(X)$ and $N^*(X)$ can be considered as simpliial sets with (1)-filtrations$\{N(X)^n\}$ and $\{N^*(X)^n\}$,where $N(X)^n = N(X^n),\ N^*(X)^{-n} = (N(X)/N(X^n))^*,\ n \ge 0$.In [7] is given that the normalized cochain complex $N(X)$ for any simplicial set X has the structure of E_∞-coalgebra $(N(X), \upsilon)$, such that the map $\upsilon(\cup_0) = \nabla : N(X) \to N(X) \otimes N(X)$ is an orthogonal approximation of Alexander-Oti $\nabla(x) = \sum_{i=0}^{n} \partial_{i+1}...\partial_n(x) \otimes \partial_0...\partial_0(x)$.

similarly on the normalized cochain complex $N^*(X)$, there is the structure of E_∞-algebra, such that the standard mulipilication of normalized cochain is given by : $\mu(\cup_0) = \cup : N^*(X) \otimes N^*(X) \to N^*(X)$.

Definition 10 : $O-$algebra (X, d, μ) is called $O-$algebra with filtratin $\{X^n\}$, if $\{X^n\}$is (1)-filtration of differential module (X, d) and for any $n \in Z,\ k_1, ..., k_j,\ j \ge 1$,the following holds

$$\mu\left(O(j)_n \otimes X^{k_1} \otimes X^{k_j}\right) \subseteq X^{n+k_1+...+k_j}.$$

Definition 11 : $O-$coalgebra (X, d, υ) is called $O-$coalgebra with filtratin $\{X^n\}$, if $\{X^n\}$is (1)-filtration of differential module (X, d) and for any $n \in Z,\ k \in Z$,the following holds

$$\upsilon\left(O(j)_n \otimes X^k\right) \subseteq \bigoplus_{k_1+...+k_j = n+k} X^{k_1} \otimes ... \otimes X^{k_j}.$$

The following gives an example of E_∞-coalgebra with filtration, that is given from filtrative simplicial set.

Theorem 1 *let X be an arbitrary simplicial set with filtration $\{X^n\}$, then E_∞-coalgebra of normalized chian complex $N(X)$ is E_∞-coalgebra with filtration $\{N(X^n)\}$ and E_∞-algebra of normalized chian complex $N^*(X)$ is E_∞-algebra with filtration $\{N^*(X)^{-n}\}$, where $N^*(X)^{-n} = (N(X)\,/\,N(X^n))^*$.*

Proof . we recall the concept of operad (E^Δ, π) from[7]. Let $\Delta[n]$ be the standard n−simplicial simplex , defined by free simplicial set induced by $i_n \in \Delta[n]_n$.Let also $\overline{\Delta}[n] = N(\Delta[n])$ is the normal chain complex of simplicial set $\Delta[n]$. The simmetric family $E^\Delta = \{E^\Delta(j)\}_{j\geq 1}$ is defined by $E^\Delta(j) = Hom(\overline{\Delta}[*]; \overline{\Delta}[*]^{\otimes j}), j \geq 1$,where, $\overline{\Delta}[*] = \{\overline{\Delta}[n]\}_{n\geq 0}$−is a cosimpliial normalchain complex of standard simplicial simplex and $Hom(\overline{\Delta}[*]; \overline{\Delta}[*]^{\otimes j})$ is a differential graded module of cosimplicial maps.The multiplication on operad is defined by the following formula:

$$\pi(e \otimes e_1 \otimes ... \otimes e_k) = (e_1 \otimes ... \otimes e_k) \circ e,\ e \in E^\Delta(j),\ 1 \leq i \leq k \tag{1}$$

In [7] it was shown that on the normalized singular chain complex $N(X)$ of a topological space X there is the natural structure of an E^Δ-coalgebra. This structure is defined by the morphism of operads $\upsilon{:}O^\Delta \to O^{N(X)}$ that is given by the

following formula: for an arbitrary element $e \in O(j) = Hom(\overline{\Delta}[*]; \overline{\Delta}[*]^{\otimes j})$ and each generator $x \in .N_n(X)$,

$$\alpha(e)(x) = (N(\overline{x}) \otimes ... \otimes N(\overline{x})e(i_n)),$$

where $N(\overline{x}) : \overline{\Delta}[n] \to N(X)$ is the chain map induced by the simplicial map $\overline{x} : \Delta[n] \to X$,$\overline{x}(i_n) = x$ and $i_n \in \overline{\Delta}[n]_n$ is a generator of the standard simplicial simplex $\Delta[n]$.We show for any simplicial set X, the E^Δ−coalgebra $N(X)$ is E^Δ−coalgebra with fltration $\{N(X^n)\}$,where, $\{X^n\}$ is filtration of simplicial set. Therfore consider the filtration $\{\overline{\Delta}[m]^k\}$ of chain complex $\overline{\Delta}[m]$,induced the filtration $\{\Delta[m]^k\}$ of standard simplicial n−simplex $\Delta[m]$. Since for every $k \geq 0$, we have $\overline{\Delta}[m]_k = \overline{\Delta}[m]^k_k$, then for any $x \in N_m(X^k) = N_m(X^m)$, $0 \leq m \leq k$ and any $e \in E^\Delta(j)_n$ we have:

$$\upsilon(e)(x) = N(\overline{x})^{\otimes j} e(i_m) \in \bigoplus_{k_1+...+k_j=m+n} N(\overline{x})(\overline{\Delta}[m]^{k_1}_{k_1}) \otimes ... \otimes N(\overline{x})(\overline{\Delta}[m]^{k_j}_{k_j} \subseteq$$

$$\underset{k_1+...+k_j=m+n}{\oplus}(N(X^{k_1})\otimes ...\otimes N(X^{k_j}))_{n+m}\subseteq \underset{k_1+...+k_j=k+n}{\oplus}(N(X^{k_1})\otimes ...\otimes N(X^{k_j}))_{n+m}$$

That is the $E^{\Delta}-$coalgebra $N(X)$ is $E^{\Delta}-$coalgebra with fltration $\{N(X^n)\}$.

From [7] we know that , for given operad E^{Δ} ,the operad morphism φ_∞ : $E_\infty \to E^{\Delta}$ satisfies the condition $\varphi_\infty(\cup_0)=\nabla$, where $\nabla : \overline{\Delta}[*]\to\overline{\Delta}[*]\overline{\otimes\Delta}[*]$ is the standard assoiative co multiplication on cosimplicial complex $\overline{\Delta}[*]$. If we take the composition operad morphism $\varphi_\infty : E_\infty \to E$ and the structure of operad morphism $\upsilon : \varphi_\infty : E^{\Delta}\to O^{N(X)}$, we get that the E_∞-coalgebra of normalized of chain complex $N(X)$ is E_∞-coalgebra with filtration $\{N(X^n)\}$ .Hence E_∞ -coalgebra of normalized of chain complex $N^*(X)$ is E_∞-algebra with filtration $\{N^*(X)^{-n}\}$.

Note that other example of E_∞-(co)algebra with filtration can be got by considering the simplicial set maps, for instant, simplicial bundle by means of Kana [2]. It is easy to prove the following assertions.

Theorem 2 *Suppose that $f : X\to Y$ is any simplicial sets morphism , $\{Y^n\}$ is the filtration of simplicial set Y, then E_∞-coalgebra of normalized chian complex $N(X)$ is E_∞-coalgebra with filtration $\{N(X^n)\}_{n\geq 0}$, where $N(X)^n = N(f^{-1}(Y^n))$. and E_∞-algebra of normalized chian complex $N^*(X)$ is E_∞-algebra with filtration $\{N^*(X)^{-n}_{n\geq 0}\}$.*

Corollary 3 *Suppose that $p : E\to B$ is any simplicial bundle by means of Kana and $\{B^n\}$ is the filtration of base B^n of this simplicial bundle, then the E_∞-coalgebra of normalized chian complex $N(E)$ of total space E of bundle $p : E\to B$ is $E_\infty-$coalgebra with filtration $\{N(E^n)\}_{n\geq 0}$, where $N(E)^n = N(p^{-1}(B^n))$.*

Corollary 4 *Suppose that $p : E\to B$ is any simplicial bundle by means of Kana and $\{B^n\}$ is the filtration of base B^n of this simplicial bundle, then the E_∞-algebra of normalized chian complex $N(E)$ of total space E of bundle $p : E\to B$ is $E_\infty-$algebra with filtration $\{N^*(E)^{-n}\}_{n\geq 0}$, where $N(E)^{-n} = (N(E)\,/\,N(E)^n(B^n))^*$.*

References

[1] Laray J. L,anneau, L'anneau spectial et L-anneau filterd'homologie d'une espace localement compact et, d'nue application continue // J. Math. Appl.1950. N. 29. P. 1-139.

[2] Serre J. P. Homologue singuliere des espaces fibres // Ann. Math.1951.Vol. 54,no. 2.P. 425505.

[3] Loday J-L and Vallette B., Algebraic operad' Springer Link, 2012.

[4] Lapin S. V., D_∞-differential E_∞-algebras and spectral sequences of fibrations,Mat. Sbornik 198:10(2007), 13791406.

[5] Smirnov, V. A. Simplicial and operad methods in algebraic topology, Factorial Press, 2002.

[6] Smirnov, V. A., On Cochain copmlex of topological space, Mat.spornik (1981), 115, No. 1, 146-158.

[7] Smirnov, V. A. Homotopy of coalgebra, Ezv. AH CCCP, Ser. Mat.(1985)-T.49, No. 6, 1302-1321.

Received: November, 2012