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On Bézout Rings

Fatima Cheniour

Department of Mathematics, Faculty of Science and Technology of Fez,
Box 2202, University S. M. Ben Abdellah Fez, Morocco
cheniourfatima@yahoo.fr

Abstract

In this paper, we study the transfer of Bézout rings to trivial ring extensions and provide a new class of Bézout rings.

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1 Introduction

All rings considered below are commutative with unit and all modules are unital. Recall that a ring R is called Bézout if every finitely generated ideal I of R is principal.

Let A be a ring, E be an A-module and $R := A \propto E$ be the set of pairs (a, e) with pairwise addition and multiplication given by: $(a, e)(b, f) = (ab, af + be)$. R is called the trivial ring extension of A by E. Recall that a prime ideal of R has always the form $Q \propto E$, where Q is a prime ideal of A [3, Theorem 25.1]. Considerable work, part of it summarized in Glaz's book [2] and Huckaba's book [3], has been concerned with trivial ring extensions. These have proven to be useful in solving many open problems and conjectures for various contexts in (commutative and non-commutative) ring theory. See for instance [2, 3, 5].

Our aim in this paper is to construct a new class of Bézout rings. For this purpose, we investigate the possible transfer of the Bézout property to trivial extension constructions and homomorphic image.

2 Main Results

This section explores trivial ring extensions of the form $R := A \propto B$, where $A \subseteq B$ is an extension of integral domains. Notice in this context that $(a, b) \in$

R is regular if and only if $a \neq 0$. The main result (Theorem 2.1) examines the transfer of Bézout property to R and hence generates new examples of Bézout rings and non-Bézout rings.

Theorem 2.1 *Let $A \subsetneq B$ be a two domains and let $R := A \propto B$ be the trivial ring extension of A by B. Then, $R := A \propto B$ is a Bézout ring if and only if so is A and $B := qf(A)$.*

We need the following Lemma before proving Theorem 2.1.

Lemma 2.2 *Let A be a domain, $K := qf(A)$, E be a K-vector space, and $R := A \propto E$ be the trivial ring extension of A by E. Then, R is a Bézout ring if and only if A is a Bézout domain and $dim_K(E) := 1$.*

Proof. Assume that A is a Bézout domain and $dim_K(E) := 1$. In this case, we may assume that $E := K$. Our aim is to show that R is a Bézout ring.

Indeed, let J be a finitely generated proper ideal of R. Set $I := \{a \in A/(a, e) \in J$ for some $e \in K\}$. We consider two cases.
Case 1: $I := 0$. Necessarily, $J := 0 \propto (1/b)L$ for some $b \neq 0 \in A$ and some finitely generated proper ideal L of A. Further, $L := Aa$ since A is a Bézout domain. Hence $J = 0 \propto A(a/b) := R(0, a/b)$, as desired.
Case 2: $I \neq 0$. Let $(a, e) \in J$ such that $a \neq 0$. Then, $(a, e)(0 \propto K) := 0 \propto K \subseteq J$; equivalently, $J := I \propto IK := I \propto K$. But $I = Aa$ for some $a \in A$ since A is a Bézout domain. Therefore, $J := I \propto K := R(a, 0)$, completing the proof.

Conversely, assume that R is a Bézout ring. We claim that A is a Bézout domain. Indeed, let I be a finitely generated proper ideal of A. Then $J := I \propto IE := I \propto E$ is a finitely generated ideal of R. So $J := R(a, e)$ for some $a \in A$ and $e \in E$. Therefore, $I := Aa$ and hence A is a Bézout domain.

To complete the proof, it suffices to show that if $dim_K(E) \geq 2$, then R is not a Bézout ring.
Assume that $dim_K(E) \geq 2$. Let $a, b \in E$ such that $\{a, b\}$ is a K-linearly independent set and set $I := R(0, a) + R(0, b)$. We claim that I is not a principal ideal. Deny. Then, there exists $c \in E$ such that $I := R(0, c)(:= 0 \propto Kc)$. Hence, $Ka + Kb = Kc$ and so $\{a, b\}$ is a K-linearly dependent set, a

contradiction. Therefore, I is not a principal ideal which means that R is not a Bézout ring and this completes the proof of Lemma 2.2.

Proof of Theorem 2.1. Assume that A is a Bézout ring and $B := qf(A)$. Then $R := A \propto B$ is a Bézout ring by Lemma 2.2.
Conversely, assume that $R := A \propto B$ is a Bézout ring. By Lemma 2.2, it remains to show that if $qf(A) := B$.

Deny. Then $qf(A) :\neq B$. This means that there exists $m \in A - \{0\}$ such that m is not invertible in B. Our aim is to show that the ideal $R(m,0)+R(0,1)$ is not principal. Deny. Then there exists $(a,b) \in R$ such that $R(m,0) + R(0,1) := R(a,b)$. We may assume that $a := m$ since $Am + A0 := Aa$.
Since $(0,1) \in R(m,0) + R(0,1) := R(a,b)$, then there exists $(c,d) \in R$ such that $(0,1) := (m,b)(c,d) = (mc, md + bc)$. Hence $c = 0$ (since $m \neq 0$) and so $md = 1$ (where $d \in B$), a contradiction since m in not invertible in B.
Therefore, $qf(A) := B$ and this completes the proof of Theorem 2.1.

Now, we are able to construct a new class of Bézout ring and a new class of non-Bézout ring.

Example 2.3 *Let Z be the ring of integers and let $Q := qf(Z)$. Then $Z \propto Q$ is a Bézout ring (by Theorem 2.1).*

Example 2.4 *Let Z be the ring of integers, $Q := qf(Z)$ and let R be the real numbers. Then $Z \propto R$ is a non-Bézout ring (by Theorem 2.1).*

Example 2.5 *Let Z be the ring of integers. Then $Z \propto Z$ is a non-Bézout ring (by Theorem 2.1).*

Next, we end this paper by giving a new class of non-Bézout rings; namely, the trivial ring extension of a local domain (A, M) by an A-module E such that $ME := 0$.

Remark 2.6 *Let (A, M) be a local domain, $E(\neq 0)$ an A-module with $ME = 0$, and let $R := A \propto E$ be the trivial ring extension of A by E. Then, R is never a Bézout ring.*

Indeed, assume that R is a Bézout ring. Let $a(\neq 0)$ be a non-invertible element of A, $e \in E - \{0\}$, and set $J := R(a,0) + R(0,e)$. Then, $J := R(b,x)$ for some $(b,x) \in J$ since R is a Bézout ring. Hence, $Aa := Ab$ and so $b := au$ for some invertible element u of A. Therefore, $R(b,x) := R(au,x) := R(u,0)(a,u^{-1}x) = R(a,u^{-1}x)$ and so we may assume that $b := a$. Then, $J := R(a,x) := R(a,0) + R(0,e)$. Since $(a,0) \in J$, there exists $(c,f) \in R$ such that $(a,0) := (c,f)(a,x) := (ca,cx)$; hence $ca := a$ in A and $cx := 0$ in E and so $c := 1$ and $x := 0$ in E which means that $J := (Aa) \propto 0$ (since $aE := 0$) and so $e := 0$ (since $(0,e) \in J$), a contradiction since $e \neq 0$. Hence, R is never a Bézout ring.

Now, we are able to construct a non-Bézout ring.

Example 2.7 *Let $A := K[[X]]$ be a local domain with maximal ideal $M := XA$, $E := A/M = K$ and let $R := A \propto E$ be the trivial ring extension of A by E. Then, R is not a Bézout ring by the above remark.*

Our next result establishes the transfer of FFP property to a particular homomorphic image.

Proposition 2.8 *Let R be a ring and let I be a finitely generated ideal of R. If R is a Bézout ring, then so is R/I.*

Proof. Let J/I be a finitely generated ideal of R, then J is a finitely generated (using the exact sequence: $0 \longrightarrow I \longrightarrow J \longrightarrow J/I \longrightarrow 0$ where I and J/I are a finitely generated R-modules). Since R a Bézout ring, then J is a principal ideal and so J/I is a principal ideal of R/I. Hence R/I is a Bézout ring.

Corollary 2.9 *Let A be a ring, E be a finitely generated A-module, and let $R = A \propto E$ be the trivial ring extension of A by E. If R is a Bézout ring, then so is A.*

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