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On Derivations of BCC-algebras

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Abstract

In this paper, the notions of left-right (resp. right-left) t-derivations of BCC-algebras are studied and some properties on t-derivations of BCC-algebras are investigated. This paper also considers t-regular t -derivations and the d_t-invariant on ideals of BCC-algebras.

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1 Introduction

Imai and Iséki [11] defined a class of algebras of type (2,0) called BCK-algebras which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra. The class of all BCK-algebras is a quasivariety. Iséki posed an interesting problem (solved by Wroński [16]) whether the class of BCK-algebras is a variety. In connection with this problem, Komori [10] introduced a notion of BCC-algebras, and Dudek [4] redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Komori. Dudek and Zhang [6] introduced a new notion of ideals in BCC-algebras and described connections between such ideals and congruences. On the other hand, Jun and Xin [9] applied the notion of derivations in ring and near-ring theory to BCI-algebras, and they also introduced a new concept called a regular derivation in BCI -algebras. They investigated some of its properties, defined a d -derivation ideal and gave conditions for an ideal to be d-derivation.Two years later, Hamza and Al-Shehri [8] studied derivation in BCK-algebras. In [9], Hamza and Al-Shehri defined a left derivation in BCI-algebras and investigated a regular left derivation. In [17, 18], the notions of derivations of weak BCC-algebras were studied and some related properties were also investigated. In this paper, we introduce the notion of t -derivation on BCC-algebras and obtain some of its related properties and we characterized Ker d by t-derivations .

2 Preliminaries

We review some definitions and properties that will be useful in our results.

Definition 2.1. Let X be a set with a binary operation "$*$" and a constant 0. Then $(X, *, 0)$ is called a BCC-algebra if the following axioms satisfied for all $x, y, z \in X$:

(i) $((x * y) * (z * y)) * (x * z) = 0$,
(ii) $0 * x = 0$,
(iii) $x * 0 = x$,
(iv) $x * x = 0$,
(v) $x * y = 0$ *and* $y * x = 0 \Rightarrow x = y$.

Define a binary relation $\leq$ *on* X *by letting* $x * y = 0$ *if and only if* $x \leq y$. *Then*$(X, \leq)$ *is a partially ordered set.*

In any BCC-algebra X for all $x, y \in X$, the following properties hold:

(1)$(x * y) * x = 0$.
(2) $x \leq y$ implies $x * z \leq y * z$.
(3) $x \leq y$ implies $z * y \leq z * x$.

Any BCK-algebra is a BCC-algebra,but there are BCC-algebra which are not BCK-algebra. Note that a BCC-algebra is a BCK-algebra if and only if

satisfies (4) $(x * y) * z = (x * z) * y$,
or (5) $(x * (x * y)) * y = 0$.

Definition 2.2. A nonempty subset S of a BCC-algebra X is called subalgebra of X if $x * y \in S$ whenever$x, y \in S$. For a BCC-algebra X, we denote $x \wedge y = y * (y * x)$ for all $x, y \in x,\ x \wedge y \leq x, y$.

Definition 2.3. A BCC-algebra is said to be commutative if and only if satisfies for all $x, y \in X$,

$x * (x * y) = y * (y * x), i.e, x \wedge y = y \wedge x$.

Definition 2.4. Let X be a BCC-algebra and $\phi \neq I \subseteq X$. I is called an ideal of X if it satisfies the following conditions:

(i) $0 \in I$,
(ii) $x * y \in I$ and $y \in I$ imply $x \in I$.

Definition 2.5. Let X be a BCC-algebra. A map $d : X \to X$ is a left-right derivation (briefly,(l, r)-derivation)of X, if it satisfies the identity
$d(x * y) = (d(x) * y) \wedge (x * d(y))$ for all $x, y \in X$,
If d satisfies the identity
$d(x * y) = (x * d(y)) \wedge (d(x) * y)$ for all $x, y \in X$.
then d is a right-left derivation (briefly, (r, l)-derivation) of X. Moreover, if d is both a (l, r) and (r, l)-derivation, then d is a derivation of X.

3 Derivations in BCC-algebras

The following definitions introduce the notion of t -derivation for a BCC-algebra.

Definition 3.1. Let X be a BCC-algebra. Then for any $t \in X$, the self map $d_t : X \to X$ is called a right translation by t and is denoted by R_t if $d_t(x) = x * t$ for all $x \in X$.

Example 3.2. Let $G = \{0, 1, 2, 3, 4\}$ be a BCC-algebra with the following Cayley table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	2	0	0	1
3	3	2	1	0	1
4	4	4	4	4	0

For any $t \in X$,define a self map $d_t : G \longrightarrow G$ by

$$d_t(x) = x * t = \begin{cases} 0 & \text{if} \quad x = 0, 1, 2, 3 \\ 4 & \text{if} \quad x = 4 \end{cases}$$

Then it is easy to check that d_t is a $t-$derivation on G.(i.e. d_t is (l, r) and $(r, l) - t-$derivations)

Example 3.3. Let $G = \{0, a, b\}$ be a BCC-algebra with the following Cayley table:

*	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

For any $t \in X$, define a self map $d_t : G \longrightarrow G$ by

$$d_t(x) = x * t = \begin{cases} 0 \text{ if } x = 0, a \\ b \quad \text{ if } x = b \end{cases}$$

Then it is easily checked that d_t is both (l, r) and $(r, l) - t-$derivation of G.

Definition 3.4. A self map d_t of a BCC-algebra X is said to be t-regular if $d_t(0) = 0$.

Theorem 3.5. A (l, r)-t-derivation of a BCC-algebra X is t-regular.

Proof.
$$\begin{aligned} d_t(0) &= d_t(0 * x) \\ &= (d_t(0) * x) \wedge (0 * d_t(x)) \\ &= ((0 * t) * x) \wedge (0 * (x * t)) \\ &= ((0 * t) * x) \wedge 0 \\ &= 0 * (0 * ((0 * t) * x)) \\ &= 0. \end{aligned}$$

Theorem 3.6. A (r, l)-t-derivation of a BCC-algebra X is *t-regular*.

Proof.
$$\begin{aligned} d_t(0) &= d_t(0 * x) \\ &= (0 * d_t(x)) \wedge (d_t(0) * x) \\ &= 0 \wedge ((0 * t) * x) \\ &= ((0 * t) * x) * (((0 * t) * x) * 0) \\ &= ((0 * t) * x) * ((0 * t) * x) \\ &= 0. \end{aligned}$$

Corollary 3.7. A *t-derivation* of a BCC-algebra X is *t-regular*.

Proposition 3.8. Let d_t be a self map of a BCC-algebra X. Then
i) If d_t is a (l, r)-t-derivation of X, then $d_t(x) = d_t(x) \wedge x$ for all $x \in X$.
ii) If d_t is a (r, l)-t-derivation of X, then $d_t(x) = x \wedge d_t(x)$ for all $x \in X$.

Proof. (i). Let d_t be a (l, r)-t-derivation of X, then
$$\begin{aligned} d_t(x) &= d_t(x * 0) \\ &= (d_t(x) * 0) \wedge (x * d_t(0)) \\ &= d_t(x) \wedge (x * 0) \\ &= d_t(x) \wedge x. \end{aligned}$$

(ii). Let d_t be a (r,l)-t-derivation of X ,Then

$$\begin{aligned} d_t(x) &= d_t(x*0) \\ &= (x*d_t(0)) \wedge (d_t(x)*0) \\ &= (x*0) \wedge d_t(x) \\ &= x \wedge d_t(x). \end{aligned}$$

Theorem 3.9. Let d_t be a t-regular (r,l)-t-derivation of a BCC-algebra X. Then, the following hold:

i) $d_t(x) \leq x$ for all $x \in X$.

ii) $d_t(x) * y \leq x * d_t(y)$ for all $x, y \in X$.

iii) $d_t(x * y) = d_t(x) * y \leq d_t(x) * d_t(y)$ for all $x, y \in X$.

iv)$d_t^{-1}(0) = Ker(d_t) = \{x \in X : d_t(x) = 0\}$ is a subalgebra of X.

iiv) $d_t(d_t(x)) \leq x$.

v) $d_t(x* d_t(x)) = 0$.

Proof. (i). For any $x \in X$, we have:

$$\begin{aligned} d_t(x) &= d_t(x*0) \\ &= (x* d_t(0)) \wedge (d_t(x)*0) \\ &= (x* 0) \wedge d_t(x) \\ &= x \wedge d_t(x) \\ &\leq x. \end{aligned}$$

(ii)*Since* $d_t(x) \leq x$ for all $x \in X$,then $d_t(x) * y \leq x * y \leq x * d_t(y)$and hence the proof follows.

(iii). For any $x, y \in X$, we have

$$\begin{aligned} d_t(x*y) &= (x*d_t(y)) \wedge (d_t(x)*y) \\ &= (d_t(x)*y)*((d_t(x)*y)*(x*d_t(y))) \\ &= (d_t(x)*y)*0 \\ &= (d_t(x)*y) \\ &\leq d_t(x)*d_t(y). \end{aligned}$$

(iv). Let $x, y \in Ker(d_t) \Rightarrow d_t(x) = 0 = d_t(y)$.

From (iii) , we have $d_t(x* y) \leq d_t(x) * d_t(y)$

$$\begin{aligned} &= 0*0 \\ &= 0 \end{aligned}$$

implying $d_t(x* y) \leq 0$ and so $d_t(x* y) = 0$.

Therefore, $x* y \in Ker(d_t)$.

Consequently $Ker(d_t)$ is a subalgebra of X.

(iiv) $d_t(d_t(x)) = d_t(x* t)$

$$\leq d_t(x) * d_t(t)$$

$$= (x* t) * (t * t)$$
$$= (x* t) * 0$$
$$= (x* t)$$
$$\leq x.$$

(v) $d_t(x* d_t(x)) = d_t(x) * d_t(x) = 0.$

Proposition 3.10. Let X be a BCC-algebra and d_t a t-derivation. If $y \in Ker(d_t)$ and $x \in X$, then $x \wedge y \in Ker(d_t)$.

Proof. Let d_t be a (l, r)-t-derivation and $y \in Ker(d_t)$. Then we get $d_t(y) = 0$,and so

$$\begin{aligned} d_t(x\wedge y) &= d_t(y * (y *x)) \\ &= (d_t(y) * (y *x)) \wedge (y * d_t(y *x)) \\ &= (0 * (y *x)) \wedge (y * d_t(y *x)) \\ &= 0 \wedge (y * d_t(y *x)) \\ &= (y * d_t(y *x)) * ((y * d_t(y *x)) * 0) \\ &= (y * d_t(y *x)) * (y * d_t(y *x)) \\ &= 0. \end{aligned}$$

Hence we have $x \wedge y \in Ker(d_t)$.

Similarly, we can prove in case of (r, l)-t-derivation.

Proposition 3.11. Let X be a commutative BCC-algebra and d_t a t-derivation.If $x \leq y$ and $y \in Ker(d_t)$, *Then* $x \in Ker(d_t)$.

Proof. Let $x \leq y$ and $y \in Ker(d_t)$. Then we get $x * y = 0$ and $d_t(y) = 0$, and so

$$\begin{aligned} d_t(x) &= d_t(x* 0) \\ &= d_t(x* (x * y)) \\ &= d_t(y * (y *x)) \\ &= d_t(y) * (y * x) \\ &= 0 * (y *x) \\ &= 0 \end{aligned}$$

Hence we have $x \in Ker(d_t)$.

Proposition 3.12. Let X be a BCC-algebra and d_t a t-derivation. If $x \in Ker(d_t)$, we have $x* y \in Ker(d_t)$ for all $y \in X$.

Proof. Let $x \in Ker(d_t)$. Then $d_t(x) = 0$. Thus we have

$d_t(x * y) = d_t(x) * y = 0 * y = 0$

which implies $x* y \in Ker(d_t)$.

Proposition 3.13. Let X be a commutative BCC-algebra and d_t a t-derivation.Then $Ker(d_t)$ is an ideal of X.

Proof. Clearly, $d_t(0) = 0$.

Let $x* y \in Ker(d_t)$ and $y \in Ker(d_t)$. Then we get
$d_t(x * y) = 0$ and $d_t(y) = 0$, and so

$$\begin{aligned} d_t(x) &= d_t(x* 0) \\ &= d_t(x* (x * y)) \\ &= d_t(y * (y * x)) \\ &= d_t(y) * (y * x) \\ &= 0 * (y * x) \\ &= 0 \end{aligned}$$

$\Rightarrow x \in Ker(d_t)$

Definition 3.14. Let d_t be a t-derivation of a BCC-algebra X . An ideal A of X is said to be d_t-invariant if $d_t(A) \subseteq A$, where $d_t(A) = \{d_t(x)|x \in A\}$.

Theorem 3.15. Let d_t be a t-derivation of a BCC-algebra X. Then every ideal A of X is d_t-invariant.

Proof. Let A be an ideal of a BCC-algebra X.
Let $y \in d_t(A)$. Then $y = d_t(x)$ for some $x \in A$.
It follows that $y * x = d_t(x) * x = 0 \in A$, which implies $y \in A$.
Thus $d_t(A) \subseteq A$. Hence A is d_t-invariant.

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