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Generalization of Generalized Supplemented Module

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Abstract. In this paper we present the second generalization of generalized supplemented module. This generalization is done through two stages namely, the transition from supplemented into Rad-supplemented followed by transition from Rad-supplemented into weakly Rad-supplemented. If R be a Bass ring and M be a injective module then M is weakly Rad-supplemented module. Also we will generalize the $\bigoplus$-supplemented module into weak-$\bigoplus$-supplemented. We prove that every lifting module is a weak-$\bigoplus$-supplemented module. These stages depend on the conditions of the modules and rings.

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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper all rings have the identity and modules are considered to be right modules. From [3], a submodule N of M is called small in M ($N \ll M$) if for every submodule of M, $N + L = M$ straight $L = M$. The dual of small is big (essential). This means any submodule N of M is big in M if the intersection of N with L is not equal to zero, where L is submodule of M. A submodule N of M is called a supplement of K in M if $N + K = M$ and N is minimal with respect to this property [15]. A module M is called supplemented if any submodule N of M has a supplement in M and M is called amply supplemented module if for any two submodules H and G with $H + G = M$, G contains a supplement of H in M [14]. Therefore if any module M has no maximal submodule this means $M = Rad(M)$ such that $Rad(M)$ is intersection of all maximal submodules of M. A module M is called lifting if for all N submodule of M, there is a decomposition $M = H \bigoplus K$ such that H submodule of N and $N \bigcap H$ is small in M [13]. Any ring R is called a left Bass ring if, $Rad(M)$ is small in M such that M≠0 [7]. A module M is called semi-local if ($M/(Rad(M))$) is semi-simple [15]. From [11] A module M is called hollow if every proper submodule N of M is small in M, and M is called hollow-lifting if every submodule N of Msuch that (M/N) is hollow has a coessential submodule that is direct summand of M. A module M is called coatomic if every proper submodule of M is contained in a maximal submodule of M[12]. A module M is called local if M has largest submodule (i.e. a proper submodule which contains all other proper submodules [15]. A ring R is local if and only if (R_R) is local module. Let M be an R-module. Any submodules N, K of M we say that N is called a weak supplement of K in M if $N + K = M$ and $N \bigcap K$ small in M. A module M is called weakly supplemented if every submodule of M has a weak supplement in M. Let N and K are submodules of M then N is called weak Rad-supplement of K in M if, $N + K = M$ and $N \bigcap K \subseteq R\text{ad}(M)$. A module M is called weakly Rad-supplemented if every submodule of M has a weak Rad-supplemented in M [7]. In this work we are going to get a new conditions to generalize supplemented module into weakly Rad-supplemented and $\bigoplus$-supplemented into weak $\bigoplus$-supplemented module. Also, we used many type of modules to satisfying this objective as local, hollow, semi-local module.

2. Weakly Rad-supplemented module

Firstly, we begin the first stage in order to get a generalization for supplemented module. Every hollow module is Rad-supplemented and a module M is hollow if and only if it local module therefore any module M is local this means M is Rad-supplement.

Definition 2.1. Let M be an R-module and N, L are submodules of M. Then L is a radical supplement (Rad-supplement) of N in M if $N + L = M$ and $N \bigcap L \subseteq R\text{ad}(L)$.

Therefore, by this definition we can say that M is Rad-supplemented if every submodule in M has a Rad-supplement.

Lemma 2.2. *Let R be a ring and let M be an injective module. Then M has no maximal submodule.*

Lemma 2.3. *Let R be a ring and M be an injective module. Then Rad(M)=M. [6].*

Remark 2.4. If $R\text{ad}(M)$=M then M is Rad-supplemented module.

Theorem 2.5. *Let R be any ring and let M be an injective module with every N, L are submodules of M and $M = L + N$, $N \cap L \ll N$. Then M is a Rad-supplemented module.*

Proof. We have L, N are submodules of M such that $M = L+N$ and $L \cap N \ll N$. Then N is supplement of L in M. If every submodule of M has supplement then M is supplemented. Now since M is injective module then by (Lemma 2.3) M=Rad(M) and by (Remark 2.4) we get M is a Rad-supplemented module. □

Remark 2.6. The transition from supplemented into Rad-supplemented it called generalization of supplemented module.

Definition 2.7. Let M be an R-module. Then M is called a projective cover of N if M is a projective and there exists an epimorphism f from M into N such that kernel of f is small in M.

Lemma 2.8. *Let $N \rightarrow M$ be a projective cover, and let N be a hollow module. Then M is a hollow module [14].*

Theorem 2.9. *Let M be an R-module. If M is a hollow module, then M is Rad-supplemented module.*

Proof. Since M is hollow module then every proper submodule of M is small in M. Therefore every submodule of M is supplement in M. Hence M is Rad-supplemented module. □

Lemma 2.10. *Let M be an R-module. If M is local module then M is a hollow module [16].*

Remark 2.11. We recall that a supplemented module is Rad-supplemented.

In order to obtain a supplemented module for Rad-supplemented we need the following condition : (A ring R is called left Bass if Rad(M) is small in M such that $M \neq 0$).

Theorem 2.12. *Let R be a left Bass ring. Then every Rad-supplemented module is supplemented [7].*

Theorem 2.13. *Let R be a Bass ring and let M be an injective module. Then M is supplemented module.*

Proof. Since M is an injective module then M has no maximal submodule. Therefore Rad(M)= M, and then M is Rad-supplemented module. Also, every Rad-supplemented module over Bass ring is supplemented module (Theorem 2.12).

□

The next stage is a generalization of Rad-supplemented module into weakly Rad-supplemented module in several ways:

Remark 2.14. Weakly supplemented module is weakly Rad-supplemented module

Now from [4] we have the following implication

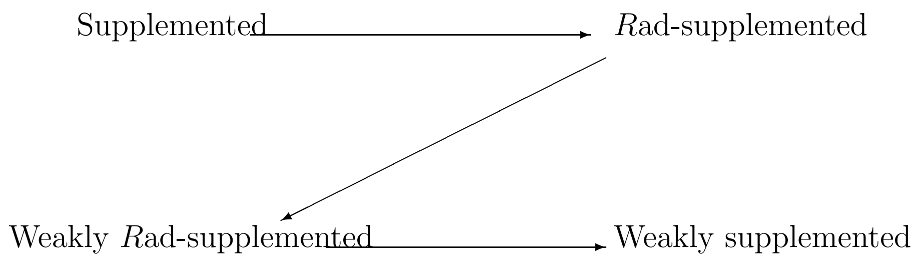

Diagram 1

There is a natural question to determine the conditions of module to transition by direct from supplemented to weakly Rad-supplemented module. We conclude:

Theorem 2.15. *Let M be a nonzero R-module. If M is local module then M is weakly Rad-supplemented module.*

Proof. : Since M is local module then by (Lemma 2.10) M is hollow module and from (Theorem 2.9) M is Rad-supplemented module. Therefore M is weakly Rad-supplemented module. □

Proposition 2.16. *Let M be an R-module and M be a coatomic with radical equal zero. If M is supplemented module then M is weakly Rad-supplemented module.*

Proof. Since M is a supplemented module then every submodule N of M is supplement of K in M such that K submodule of M. Let (M/N) be hollow. Now we have $Rad(M)$=0, then $M = K + N$, therefore M is hollow-lifting [see 13] and hence is hollow with $Rad(M) \neq M$ this means M is hollow and finitely generated (cyclic). If M is hollow with cyclic then M is local, therefore M is weakly Rad-supplemented module (Theorem 2.15). □

Now we can return to (Theorem 2.13) to obtain weakly Rad-supplemented module:

Theorem 2.17. *Let R be a Bass ring and M be injective module. If every submodule N of M such that Rad(N) $\subseteq$ Rad(M) then M is weakly Rad-supplemented module.*

Proof. By (Theorem 2.13) M is supplemented module. Then every supplemented module is Rad-supplemented module this means $L + N = M$ and $N \cap L \subseteq R\text{ad}(N)$, but $R\text{ad}(N) \subseteq R\text{ad}(M)$ then $N \cap L \subseteq R\text{ad}(M)$ therefore N is weak Rad-supplemented and hence M is weakly Rad-supplemented module. □

Lemma 2.18. *Let M be an R-module. Then M is finitely generated supplemented module if and only if $M = L_1+L_2+...+L_n$, for some local module L_n.*

Proof. See [16]. □

Theorem 2.19. *Let M be an R-module.If M is finitely generated and supplemented,then M is a weakly Rad-supplemented module.*

Proof. Since R is semi-perfect then if every finitely generated R-module has a projective cover. From[15], every finitely generated left (respectively, right) R-module is supplemented and hence M is a weakly Rad-supplemented. □

Definition 2.20. Let N submodule of M. Then N is lies over a summand of M if there is a direct decomposition $M= R \bigoplus$S with $R \subseteq N$, $S \cap N$ submodule of M.

Proposition 2.21. *Let M be a nonzero module and let ψ be homomorphism from M into $(M/(Rad(M)))$, and every submodule of M lies over a summand of M. Then M is weakly Rad-supplemented module.*

Proof. Let N be a small in $(M/(R\text{ad}(M)))$, then there is a K submodule of M such that $\psi(K)= N$. Since every submodule of M lies over a summand of M, then there exists submodules Q,R of M such that $M= Q \bigoplus R$,$Q \leq K$ and $K \bigcap R$ is small in M. Then $\psi(Q)= \psi(K)=N$ and $(M/(R\text{ad}(M)))\psi=(Q)\bigoplus \psi(R)=N \bigoplus \psi(R\,)$. N is direct summand of $(M/(R\text{ad}(M)))$.Then $(M/(R\text{ad}(M))$) is semi-simple. If $(M/(R\text{ad}(M)))$ is semi-simple then M is semi-local and so R semi-simple which means R semi-perfect and by (Theorem 2.20) M is a weakly Rad-supplemented module. □

Theorem 2.22. *Let M be an R-module such that M not equal $Rad(M)$. Suppose that for every N submodule of M, there is a direct decomposition M=$A \bigoplus B$ with $A \subseteq N$,$B \cap N$ submodule of M and indecomposable. Then M is a weakly Rad-supplemented module.*

Proof. Since M is indecomposable then M is local module [10] and by (Theorem 2.15) M is weakly Rad-supplemented module. □

Theorem 2.23. *Let N be submodule of a module M. If M is indecomposable and $Rad(M) \ll M$ and if M is N-semi-potent,then M is a weakly Rad-supplemented module.*

Proof. By [1] M is local module and by (Theorem 2.15) M is weakly Rad-supplemented module. □

Proposition 2.24. *Let M be an R-module. If there exists a maximal submodule H of M such that H is small in M, then M is local and then weakly Rad-supplemented module.*

Proof. Let K be a proper submodule of M. Suppose that K not equal M. Then H $\subseteq H + K \subseteq M$. Therefore $H = H + K$ or $H + K = M$. Let $H + K = M$, which means $K = M$. This contradiction with H small in M, therefore, $H = H + K$, then $K \subseteq H$ and hence H is largest proper submodule in M (M is local). Then M is weakly supplemented, and hence M is weakly Rad-supplemented module. □

The most important results can be summarized in the following diagrams:

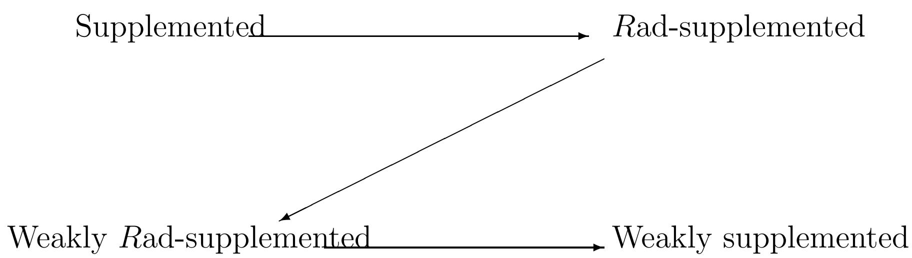

Diagram 2

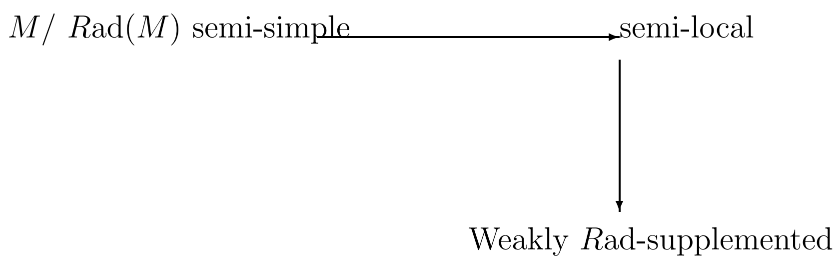

Diagram 3

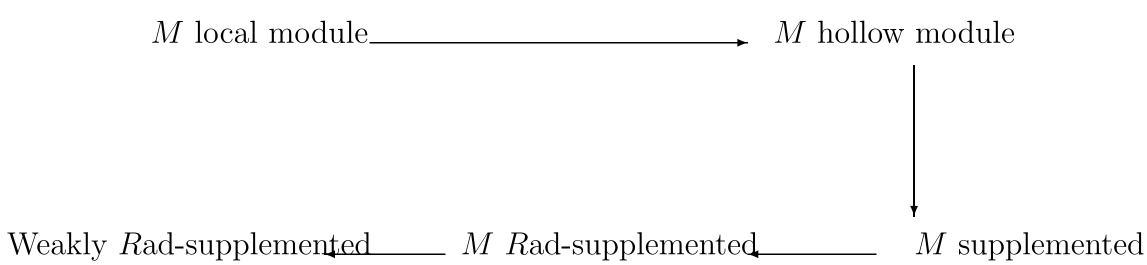

Diagram 4

3. Weak-$\bigoplus$-supplemented module

In this section we are studying some properties of $\bigoplus$-supplemented module and make a generalization of $\bigoplus$-supplemented into weak $\bigoplus$-supplemented by new conditions. If M lifting module then M is a $\bigoplus$-supplemented module. Also if R be a semi-simple ring and M be a $\bigoplus$-supplemented module then M is injective module. Let R be a ring then if every $\bigoplus$-supplemented R-module is injective this imply R is a left Noetherian V-ring and semi-simple [2].

Definition 3.1. Let M be an R-module. If any submodule of M has a supplement that is a direct summand of M and M is supplemented, then M is called a $\bigoplus$-supplemented module.

Definition 3.2. A module M is called amply supplemented if for any two submodules H and K with $H + K = M$, K contains a supplement of H in M.

Now we can rewrite above definition by other way in order to obtain $\bigoplus$-supplemented module. We explore many new properties for $\bigoplus$-supplemented module. If M is amply supplemented and any supplement submodule of M is a direct summand of M then M has lifting property. Therefore we can present the following lemmas:

Lemma 3.3. *Let M be an R-module. If M is lifting module then M is a $\bigoplus$-supplemented module.*

Lemma 3.4. *Let N be submodule of M, then there exists a direct summand L of M such that L submodule of N and (N/L) submodule of Rad $((M)/(L))$. Then M is hollow and $\bigoplus$-supplemented module.*

Definition 3.5. Let M be an R-module and N submodule of M. Then N lies above a direct summand of M if there exists H and G are submodules of N such that $H \bigoplus G = M$ and $N \cap G \ll G$.

Definition 3.6. Any module M is called (D_1)-module if every N submodule of M is lies above a direct summand of M.

Lemma 3.7. *Let M be an R-module. Then M is (D_1)-module if and only if M is lifting module.*

Theorem 3.8. *Let M be an R-module. If M is (D_1)-module, then M is a $\bigoplus$-supplemented module.*

Proof. Since M is (D_1)-module, then N is lies above a direct summand of M. Therefore there exists H, G are submodules of N, such that $H \bigoplus G = M$ and $N \cap G \ll G$. Then M is a lifting module and by (Lemma 3.3) M is a $\bigoplus$-supplemented module. □

Corollary 3.9. *Let M be an amply supplemented module such that every submodule of M lies above a direct summand of M. Then M is a $\bigoplus$-supplemented module.*

Proof. Suppose M is amply supplemented module. Then there are submodules L, K of M such that $L + K = M$ and K contains supplement of L. Since every submodule of M lies above a direct summand of M then M is (D_1)-module therefore$M$ is a lifting module and hence by (Lemma 3.3) M is a $\bigoplus$-supplemented module. □

Corollary 3.10. *If M is a (D_1)-module, then $(M/Rad(M))$is a $\bigoplus$-supplemented module.*

Proof. Let M be (D_1)-module. Then every submodule of M lies above a direct summand of M this means M is a lifting module. Then M is a$\bigoplus$-supplemented module and hence $(M/(R\text{ad}(M)))$ is a $\bigoplus$-supplemented[8]. □

Proposition 3.11. *Let M be an R-module and let N submodule of M such that N supplement in M. Then M is a $\bigoplus$-supplemented module*

Proof. Firstly, we must prove that $R\text{ad}(M)$=0. Suppose $R\text{ad}(M)$ is not equal zero, there exists nonzero element r belong to $R\text{ad}(M)$. We have Rr supplement, then $Rr + H = M$ and $Rr \cap H \ll Rr$ such that H submodule of M. Since r belong to $R\text{ad}(M)$, then $Rr \ll M$ and $H = M$ and hence $Rr \ll Rr$ which is impossible. Then $R\text{ad}(M)$=0. Now we have N is a supplement, then $M = N + K$ and $N \cap K \ll K$ such that K submodule of M therefore $N \cap K \subseteq R\text{ad}(M)$ =0, then $N \cap K$ =0 and hence $M = N \bigoplus K$, then M is semi simple and hence is a $\bigoplus$-supplemented module. □

Lemma 3.12. *Let R be a semi-simple ring and M be a $\bigoplus$-supplemented module. Then M has no maximal submodule[2].*

Definition 3.13. A ring R is called a left V-ring if every simple left R-module is injective [2].

Theorem 3.14. *Let R be V-ring and M be an R-module. If M is the sum of its simple submodules, then M is a $\bigoplus$-supplemented module.*

Proof. Since M is the sum of simple submodules then M is semi-simple [15]. Now M is semi-simple and R is V-ring. Then M is a $\bigoplus$-supplemented module [2]. □

Proposition 3.15. *Let M be an R-module. If N submodule of M is linearly compact and lies above a direct summand of M. Then M is a $\bigoplus$-supplemented module.*

Proof. Since N is linearly compact then N has ample supplements in M [15]. Therefore if K submodule of M and L submodule of N such that $N + K = M$, there is supplement L of N. Then by (Definition 3.2) M is amply supplemented module. Now M is amply supplemented and N is lies above a direct summand of M then by (Corollary 3. 9) M is a $\bigoplus$-supplemented module
. □

Definition 3.16. Let M be an R-module. Then M is weak-$\bigoplus$-supplemented module if for each semi-simple submodule N of M there exists a direct summand K of M such that $M = N + K$ and $N \cap K$ is small in K.

Remark 3.17. Note that w-$\bigoplus$-supplemented module not imply $\bigoplus$-supplemented but the converse is true.

Example 3.18. *Let R be a local Artinian ring with radical W such that $W^2 =$ 0, $Q = R = W$ is commutative, dim(QW) = 2 and dim(WQ) = 1. Then the indecomposable injective right R-module U = [($R \bigoplus R$)/D] with $D =$ (ur; -v r): $r \in R$ is a w-$\bigoplus$-supplemented module, but is not $\bigoplus$-supplemented [13].*

Theorem 3.19. *Let M be an R-module. If M is a lifting module then M is w-$\bigoplus$-supplemented module.*

Proof. Since M is a lifting module then M is amply supplemented and any supplement submodule of M is direct summand of M. Therefore by (Corollary 3.9)M is a $\bigoplus$-supplemented module and by(Remark 3.17) M is a w-$\bigoplus$-supplemented module. □

Corollary 3.20. *If M is (D_1)-module then M is a w-$\bigoplus$-supplemented module.*

Proof. Since M is (D_1)-module then M is lifting module and by above theorem M is a w-$\bigoplus$-supplemented module. □

Corollary 3.21. *Let N be submodule of M.If N lies above a direct summand of M then M is a w-$\bigoplus$-supplemented module.*

Proof. By (Definition 3. 6) and (Corollary 3.21). □

Corollary 3.22. *If M is Noetherian R-module and (D_1)-module then M is a w-$\bigoplus$-supplemented module.*

Proof. Since M is Noetherian module then every submodule of M is finitely generated. Now M is (D_1)-module that is mean M is lifting module therefore M is finitely lifting. Then M is lifting module [18] and hence M is a w-$\bigoplus$-supplemented module (Theorem 3.19.) □

Therefore we get the following implications:

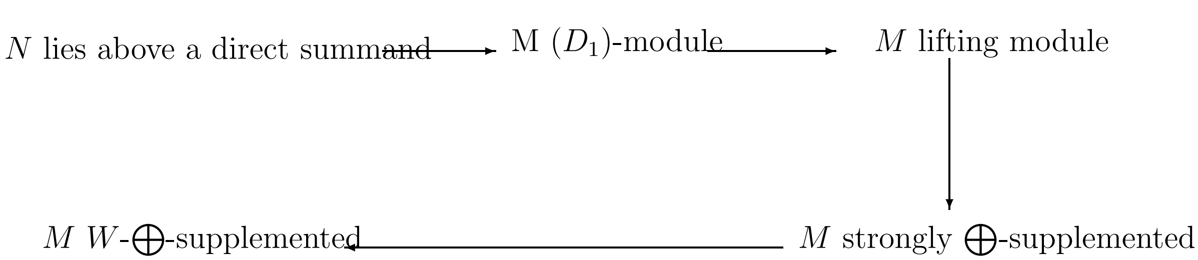

Diagram 5

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