International Journal of Algebra, Vol. 6, 2012, no. 27, 1303 - 1308

Generalization of $\oplus-\delta-$Supplemented Modules

Figen YÜZBAŞI ERYILMAZ and Şenol EREN

Ondokuz Mayıs University, Faculty of Sciences and Arts
Department of Mathematics, 55139, Samsun, Turkey
figenyuzbasi@gmail.com, seren@omu.edu.tr

Abstract

Let R be a ring and M be a left $R-$module. We say that an $R-$module M is a generalized $\oplus-\delta-$supplemented module if every submodule of M has a generalized $\delta-$supplement which is a direct summand of M. In this paper, several properties of these modules are given. We showed that any finite direct sum of generalized $\oplus-\delta-$supplemented modules is a generalized $\oplus-\delta-$supplemented module and every direct summand of a UC-extending generalized $\oplus-\delta-$supplemented module is a generalized $\oplus-\delta-$supplemented.

Mathematics Subject Classification: 16D10, 16D90, 16D99, 16N80

Keywords: $\delta-$small submodule, $\oplus-\delta-$supplemented module, generalized $\oplus-\delta-$supplemented module

1 Introduction & Preliminaries

Throughout this paper R will be an associative ring with identity and all modules will be unital left $R-$modules. Let M be an $R-$module. The notation $N \leq M$ means that N is a submodule of M. $Rad\,(M)$ will indicate Jacobson radical of M. A submodule N of an $R-$module M is called *small* in M and written $N \ll M$, if $M \neq N + L$ for every proper submodule L of M. By a supplement of N in M we mean a submodule K which is minimal in the collection of submodules L of M such that $M = N + L$. The module M is called *supplemented* if every submodule has a suplement in M. For more detailed discussion on supplemented modules we refer to [6, 15].

Following [7], a module M is called $\oplus-$*supplemented* if every submodule of M has a supplement that is a direct summand of M. $\oplus-$supplemented modules are studied in [2, 3, 4].

Let N and K be any submodules of M with $M = N + K$. If $N \cap K \leq Rad\,(K)$, then K is called *generalized supplement* (according to [1], radical supplement or briefly Rad-supplement) of N in M [13]. The notion of generalized supplemented modules was introduced by Xue in [16]. An $R-$module M is called *generalized supplemented* module or briefly $GS-$module (in [1] Rad-supplemented) if every submodule of M has a generalized supplement in M. On the other hand, the module M is called a *Rad-$\oplus$-supplemented* module and denoted by Rad-$\oplus - s-$module, if every submodule of M has a Rad-supplement that is a direct summand of M. These modules were studied in [10, 11].

In [17], Zhou introduced the concept of $\delta-$small submodules as a generalization of small submodules. Recall that a submodule N of a module M is said to be $\delta-$*small* in M and denoted by $N \ll_{\delta} M$ if whenever $M = N + K$ and$\frac{M}{K}$ singular, we have $M = K$. The sum of all $\delta-$small submodules of a module M is denoted by $\delta(M)$, which defines a preradical on the category of $R-$modules.

Let N be a submodule of a module M. A submodule K of M is called a $\delta-$*supplement* of N in M provided that $M = N + K$ and $M \neq N + X$ for any proper submodule X of K with $\frac{K}{X}$ singular; or equivalently, $M = N + K$ and $N \cap K \ll_{\delta} K$. The module M is called $\delta-$*supplemented* if every submodule of M has a $\delta-$supplement in M [5, 14]. Also, M is called $\oplus - \delta-$*supplemented* if every submodule of M has a $\delta-$supplement which is a direct summand of M [12]. Clearly $\oplus - \delta-$supplemented modules are $\delta-$supplemented and $\oplus-$supplemented modules are $\oplus - \delta-$supplemented.

According to [9], for two submodules N and K of M, N is called *generalized $\delta-$supplement* of K in M if $M = N + K$ and $N \cap K \leq \delta(N)$. The module M is called *generalized $\delta-$supplemented* or briefly $\delta - GS$ if every submodule N of M has a generalized $\delta-$supplement in M.

In this note, we introduce generalized $\oplus - \delta-$supplemented modules. We answer the following natural question: is any factor module of a generalized $\oplus - \delta-$supplemented module generalized $\oplus - \delta-$supplemented? In addition, we investigate direct summand of these modules.

2 Main Results

Definition 1 *A module M is called generalized $\oplus - \delta-$supplemented module if every submodule of M has a generalized $\delta-$supplement which is a direct summand of M.*

An $R-$ module M is said to have property (D_3), if M_1 and M_2 are direct summand of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M [15].

Proposition 2 *Let M be a generalized $\oplus-\delta-$supplemented module with (D_3). Then every direct summand of M is a generalized $\oplus-\delta-$supplemented module.*

Proof. Let U be a direct summand of M and N be a submodule of U. Then there exists a direct summand V of M such that $M = N+V$ and $N \cap V \leq \delta(V)$. By modularity, we have $U = N+(V \cap U)$. Since M has (D_3) property, $(U \cap V)$ is a direct summand of M and so it is also a direct summand of U. Note that $N \cap (U \cap V) = N \cap V \leq \delta(V)$. By Lemma 2.2 in [9], $N \cap V \leq \delta(U \cap V)$. Therefore U is a generalized $\oplus - \delta-$supplemented module. ■

Let M be a module. A submodule N of M is *closed* in M if N has not a proper essential extension in M. In [8], he called a module M is a *$UC-$module* if every submodule of M has a unique closure in M. M is called *extending module* if every closed submodule of M is a direct summand of M.

Corollary 3 *Let M be a $UC-$extending module. If M is a generalized $\oplus - \delta-$supplemented module, then every direct summand of M is a generalized $\oplus - \delta-$supplemented module.*

Proof. Since M is a $UC-$extending module, M has (D_3) by Lemma 2.4 in [2]. Therefore the result follows from Proposition 2. ■

Theorem 4 *Let M_1 and M_2 be generalized $\oplus - \delta-$supplemented modules. If $M = M_1 \oplus M_2$, then M is a generalized $\oplus - \delta-$supplemented module.*

Proof. Let K be any submodule of M. Then $M = M_1 + M_2 + K$ and so M_1+M_2+K has a generalized $\delta-$supplement 0 in M. Since M_1 is a generalized $\oplus - \delta-$supplemented module, $M_1 \cap (M_2 + K)$ has a generalized $\delta-$supplement X in M_1 such that X is direct summand of M_1. By Proposition 2.7 in [9], X is a generalized supplement of $M_2 + K$ in M. Since M_2 is a generalized $\oplus - \delta-$supplemented module, $M_2 \cap (K + X)$ has a generalized $\delta-$supplement Y in M_2 such that Y is direct summand of M_2. Again applying Proposition 2.7 in [9], we get $X+Y$ is a generalized $\delta-$supplement of K in M. Clearly, $X \oplus Y$ is a direct summand of M. Thus $M_1 \oplus M_2$ is a generalized $\oplus - \delta-$supplemented module. ■

Corollary 5 *Any finite direct sum of generalized $\oplus - \delta-$supplemented modules is a generalized $\oplus - \delta-$supplemented module.*

Let M be a module with $S = End\,({}_RM)$. A submodule N is called *fully invariant* if for each $f \in S$, $f(N) \leq N$. Also M is called *duo* provided, every submodule of M is fully invariant [7].

Proposition 6 *Let M be a nonzero generalized $\oplus - \delta-$supplemented module and U be a fully invariant submodule of M. Then the factor module $\frac{M}{U}$ is a generalized $\oplus - \delta-$supplemented module.*

Proof. For any submodule L of M containing U, let $\frac{L}{U}$ be any submodule of $\frac{M}{U}$. Since M is a generalized $\oplus-\delta-$supplemented module, there exist submodules N and N' of M such that $M = L + N$, $L \cap N \leq \delta(N)$ and $M = N \oplus N'$. By Proposition 2.9 in [9], $\frac{(N+U)}{U}$ is a generalized $\delta-$supplement of $\frac{L}{U}$ in $\frac{M}{U}$. If we use Lemma 2.4 in [3], then we get $U = (U \cap N) \oplus (U \cap N')$. It follows that $(N + U) \cap (N' + U) \leq U$ and so $\frac{M}{U} = \frac{(N+U)}{U} \oplus \frac{(N'+U)}{U}$. Then $\frac{(N+U)}{U}$ is a generalized $\delta-$supplement of $\frac{L}{U}$ such that $\frac{(N+U)}{U}$ is a direct summand of $\frac{M}{U}$. Consequently, $\frac{M}{U}$ is a generalized $\oplus-\delta-$supplemented module. ■

Corollary 7 *Let M be a generalized $\oplus-\delta-$supplemented and duo module. Then every factor module of M is a generalized $\oplus-\delta-$supplemented module.*

Proposition 8 *Let M be a generalized $\oplus-\delta-$supplemented module and U be a fully invariant submodule of M. If U is a direct summand of M, then U is a generalized $\oplus-\delta-$supplemented module.*

Proof. Let U be a direct summand of M and N be a submodule of U. Since M is a generalized $\oplus-\delta-$supplemented, there exist L and L' of M, such that $M = N + L$, $N \cap L \leq \delta(L)$ and $M = L \oplus L'$. By Lemma 2.4 in [3], we have $U = (U \cap L) \oplus (U \cap L')$. If we show that $N \cap (U \cap L) = N \cap L \leq \delta(U \cap L)$, then the proof is complete. Since $M = N + L$, we have $U = N + (U \cap L)$, $N \cap L \leq \delta(M)$. Due to $U \cap L$ is a direct summand of M, we obtain $N \cap L \leq \delta(U \cap L)$ by Lemma 2.2 in [9]. Hence $U \cap L$ is a generalized $\delta-$supplement of N in U that is a direct summand of U. So it implies that U is a generalized $\oplus-\delta-$supplemented module. ■

Theorem 9 *Let M be a module such that $M = M_1 \oplus M_2$ is a direct sum of submodules M_1 and M_2. Then M_2 is a generalized $\oplus-\delta-$supplemented module if and only if there exists a direct summand K of M such that $K \leq M_2$, $M = N + K$ and $N \cap K \leq \delta(K)$ for every submodule $\frac{N}{M_1}$ of $\frac{M}{M_1}$.*

Proof. Let $\frac{N}{M_1}$ be any submodule of $\frac{M}{M_1}$. By hypothesis, there exist K, K' submodules of M_2 such that $M_2 = (N \cap M_2) + K$, $(N \cap M_2) \cap K = N \cap K \leq \delta(K)$ and $M_2 = K \oplus K'$. Note that $M = M_1 + M_2 = N + K$. Since K is a direct summand of M_2, we have K is a direct summand of M.

Conversely, suppose that $\frac{M}{M_1}$ has the stated property. Let H be a submodule of M_2. Consider the submodule $\frac{(H \oplus M_1)}{M_1} \leq \frac{M}{M_1}$. By hypothesis, there exists a direct summand K of M such that $K \leq M_2$, $M = (H + K) \oplus M_1$ and $K \cap (H + M_1) \leq \delta(K)$. Then $M_2 = H + K$ and $H \cap K \leq \delta(K)$. Thus K is a generalized $\delta-$supplemented of H in M_2 and it is a direct summand of M_2. Therefore M_2 is generalized $\oplus-\delta-$supplemented. ■

Theorem 10 *Let M_i $(1 \leq i \leq n)$ be any finite collection of relatively projective modules. The module $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is a generalized $\oplus-\delta-$supplemented module if and only if M_i is a generalized $\oplus-\delta-$supplemented module for each $1 \leq i \leq n$.*

Proof. The necessity part is proved in Theorem 4.

Conversely, it is sufficient to prove that M_1 is generalized $\oplus-\delta-$supplemented. Let N be any submodule of M_1. Then there exist submodules K and K' of M such that $M = N + K = K \oplus K'$ and $N \cap K \leq \delta(K)$. Note that $M = N + K = M_1 + K$. By Lemma 4.47 in [6], there exists a submodule K_1 of K such that $M = M_1 \oplus K_1$. If we intersect the last equation with K_1, then we get $K = (M_1 \cap K) \oplus K_1$. Since $M_1 = (M_1 \cap K) + N$ and $M_1 \cap K$ is a direct summand of M_1, we obtain $N \cap K \leq \delta(M_1 \cap K)$ if we use Lemma 2.2 in [9]. Therefore M_1 is a generalized $\oplus-\delta-$supplemented module. ■

References

[1] J. Clark, C. Lomp, N. Vajana and R. Wisbauer, Lifting Modules, Birkhauser Verlag, 2006.

[2] A. Harmancı, D. Keskin, P.F. Smith, *On $\oplus-$supplemented modules*, Acta Math. Hungar., **83(1-2)**(1999), 161-169.

[3] A. Idelhadj and R. Tribak, *On sum properties of $\oplus$-supplemented modules*, Int. J. Math. Sci. , **69**(2003), 4373-4387.

[4] D. Keskin, P. F. Smith and W. Xue, *Rings whose modules are $\oplus$-supplemented,* J. Algebra, **218(2)**(1999), 470-487.

[5] M. T. Koşan, *$\delta-$lifting and $\delta-$supplemented modules*, Algebra Colloq., **14(1)**(2007), 53-60.

[6] S.H. Mohamed, B.J. Müller, Continuous and Discrete Modules, Cambridge University Press, 1990.

[7] A. Ç. Özcan, A. Harmancı and P.F. Smith, *Duo modules*, Glasgow Math. J., **48(3)**(2006), 533-545.

[8] P.F. Smith, Modules for which every submodule has a unique closure, in Ring Theory, World Sci., Singapore, 302-313, 1993.

[9] Y. Talebi and B. Talaee, *On generalized $\delta-$supplemented modules*, Vietnam J. Math., **37(4)**(2009), 515-525.

[10] Y. Talebi, A. R. Hamzekolaei and D. K. Tütüncü, *On Rad-⊕-supplemented modules*, Hadronic J., **32(5)**(2009), 505-512.

[11] Y. Talebi and A. R. Mahmoudi, *On Rad-⊕-supplemented modules*, Thai J. Math., **9(2)**(2011), 373–381.

[12] B. Ungor, S. Halıcıoğlu and A. Harmancı, *On a class of $\delta-$supplemented modules*, Bull. Malays. Math. Sci. Soc., (accepted)

[13] Y. Wang and N. Ding, *Generalized supplemented modules*, Taiwanese J. Math., **10(6)**(2006), 1589-1601.

[14] Y. Wang, *$\delta-$small submodules and $\delta-$supplemented modules*, Int. J. Math. and Math. Sci., (2007).

[15] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, Philadelphia, 1991.

[16] W. Xue, *Characterizations of semiperfect and perfect rings*, Publ. Mat., **40(1)**(1996), 115-125.

[17] Y. Zhou, *Generalizations of perfect, semiperfect, and semiregular rings,* Algebra Colloq., **7(3)**(2000), 305-318.

Received: August, 2012