International Journal of Algebra, Vol. 6, 2012, no. 21, 1025 - 1038

Relation between α-Hochschild Homology and α-Hochschild Cohomology of a Quantum Algebra

Eleonora Cerati

Departamento de Matematica - Facultad de Humanidades y Ciencias
Universidad Nacional del Litoral
Ciudad Universitaria-Paraje El Pozo s/n - 3000 Santa Fe - Argentina
eleonoracerati@gmail.com

Ingrid Schwer

Departamento de Matemtica - Facultad de Ingenieria Quimica
Universidad Nacional del Litoral
Santiago del Estero 2829 - 3000 Santa Fe - Argentina
ischwer@fiq.unl.edu.ar

Abstract

Using a quantum version of the Koszul complex and techniques defined by Wambst [12] which allow to build complexes quasi isomorphic to the standard Hochschild complex, we compute the Hochschild $\alpha-$homology of the algebra $A = \mathbb{C}\{x_1, \ldots, x_n\}$ generated over $\mathbb{C}$ by $x_1, \ldots, x_n$ with relations $x_i x_j = q_{ij} x_j x_i (1 \leq i, j \leq n)$ for a linear automorphism α of A.

For any multi-integer $\gamma \in \mathbb{N}^n$ the Koszul complex $K_\gamma(A, A_\alpha)$, where A_α coincides with A as a vector space but the structure of right A-module is twisted by the morphism α, splits into a direct sum of subcomplexes K_γ of elements of degree γ in the Koszul complex. Calling $C = \{\gamma \in \mathbb{N}^n / \forall i, 1 \leq i \leq n, \gamma_i = 0 \text{ or } x^\gamma x_i = \alpha(x_i) x^\gamma\}$, we prove that $HH_\alpha(A) = \oplus_{\gamma \in C} K_\gamma$.

Since the algebra A is AS-Gorenstein, generalized Koszul and of finite global dimension, according to Berger and Marconnet [1], we apply Van den Bergh Duality Theorem [10] to compute $\alpha-$Hochschild cohomology.

Mathematics Subject Classification: 16E40, 81R50

Keywords: $\alpha-$homology, $\alpha-$cohomology, quantum algebra

1 Introduction

In this work, we use techniques defined by Wambst [12], generalizing objects from commutative algebra, conveniently adapted for the computation of the cohomology of quantum algebras such as the quantum version of the Koszul complex. These methods are applied to the case of the multiparameter quantum affine space. We compute $HH^i_\alpha(A)$ for $0 \leq i \leq 4$ where $A = \mathbb{C}\{x_1, \ldots, x_n\}$ is the algebra generated over $\mathbb{C}$ by $x_1, \ldots, x_n$ with $x_i x_j = q_{ij} x_j x_i$ and α is a linear automorphism of A. Diverse results are obtained when we consider different types of automorphisms. To generalize the results obtained we compute the $\alpha-$ Hochschild homology of A using Van den Bergh Duality Theorem [10].

In section 2, we recall some definitions and propositions about the multiparameter quantum affine space, its automorphisms and α- derivations, given in [3]. We compute the $\alpha-$Hochschild cohomology groups in degrees 2, 3 and 4 for different types of linear automorphisms using the definition given by Redondo and Solotar [6] and the computation of the $\alpha-$Hochschild cohomology in degrees 0 and 1 given in [3].

In section 3 we prove a theorem that allow us to compute the $\alpha-$ Hochschild homology of the quantum algebra A using techniques defined by Wambst [12].

In section 4, we study the relation between the $\alpha-$ Hochschild homology and the $\alpha-$ Hochschild cohomology of A. Since A is a $2-$homogeneous graded algebra, AS-Gorenstein and generalized Koszul, we apply Van den Bergh Duality Theorem [10] to compute the $\alpha-$ Hochschild cohomology. We obtain that $HH^0_\alpha(A) = \mathbb{C}$ and $HH^n_\alpha(A) = 0$ for $n > 0$, when α is of *type 1*. For α of *type 2*, $HH^n_\alpha(A) = \mathbb{C} \oplus \mathbb{C} x_1^{j_1+1} x_2^{j_2+1} \ldots x_n^{j_n+1}$ for $n > 0$. This generalizes the obtained results for $n = 2, 3$ and 4 in [2] and [3].

2 Preliminaries

2.1 The multiparameter quantum affine space

Let $\mathbb{C}$ be the complex field and $Q = (q_{ij}) \in M_n(\mathbb{C})$ with $n \geq 2$, whose elements verify $q_{ii} = 1$, $q_{ij} = q_{ji}^{-1}$ if $i \neq j$ and q_{ij} is not a root of unit. Let $A = \mathbb{C}\{x_1, \ldots, x_n\}$ be the algebra generated by $x_1, \ldots, x_n$ with relations

$$x_i x_j = q_{ij} x_j x_i \text{ for } 1 \leq i, j \leq n.$$

In fact, A is the algebra of the quantum affine space of dimension n, that is a graded algebra with grading given by the total degree. The monomials $x^t = x_1^{t_1} x_2^{t_2} \ldots x_n^{t_n}$, with $t = (t_1, t_2, \ldots, t_n) \in \mathbb{N}^n$ form a basis of A as $\mathbb{C}-$vector

space. It is then verified that:

$$x_j\ x^t = \prod_{1\leq \ell<j\leq n} q_{j\ell}^{t_\ell}\ x_1^{t_1}\dots x_{j-1}^{t_{j-1}} x_j^{t_j+1} x_{j+1}^{t_{j+1}}\dots x_n^{t_n}$$

$$x^t\ x^s = \prod_{1\leq \ell<k\leq n} q_{k\ell}^{t_k s_\ell - s_k t_\ell}\ x^s\ x^t$$

where $x^s = x_1^{s_1}\dots x_n^{s_n}$.

2.2 The automorphisms of A

Let $\alpha : A \to A$ be a $\mathbb{C}-$automorphism of algebras. Since $\alpha(x_1^{i_1}\dots x_n^{i_n}) = [\alpha(x_1)]^{i_1}\dots[\alpha(x_n)]^{i_n}$, α is uniquely determined by $\alpha(x_i)$, $1\leq i\leq n$.

As it has been proved in [3], the automorphisms obtained are of type

$$\alpha(x_i) = a_i x_i + f_i$$

where $a_i \in \mathbb{C} - \{0\}$ and $gr(f_i) \geq 2$ or $f_i = 0$.

From now on, we will work with linear automorphisms, that is $f_i = 0$.

2.3 The $\alpha-$centralizer

The $\alpha-$centralizer of the generators of A is:

$$\begin{aligned} Z^\alpha(\{x^s\}) &= \{x^j / x^j x^s = \alpha(x^s) x^j, j\in\mathbb{Z}^n\} \\ &= \{x^j / \prod q_{k\ell}^{j_k s_\ell - j_\ell s_k} x^s x^j = a(x^s) x^s x^j, 1\leq \ell<k\leq n\}, \end{aligned}$$

where $a(x^s) = a_1^{s_1}\dots a_n^{s_n}$.

Remark 2.1 *Let $\mathbb{C}\{x_1,\dots,x_n\}$ be the quantum affine space of dimension n, with $q_{ij}\in\mathbb{C}$ and let $|q_{ij}|>1, (1\leq i<j\leq n)$. Then $Z^\alpha(\{x^s\}_{s\in\mathbb{Z}^n}) \neq \emptyset$ if and only if there are integers $j_1, j_2, \dots, j_n$ such that $\alpha(x_i) = \prod_{k\neq i} q_{ki}^{j_k} x_i$. In this case $Z^\alpha(\{x^s\}_{s\in\mathbb{Z}^n}) = \left\{x_1^{j_1} x_2^{j_2}\dots x_n^{j_n}\right\}$.*

2.4 Classification of linear automorphisms

The above remark suggests the following classification of the linear automorphisms: the automorphisms $\alpha \in Aut\ \mathbb{C}\{x_1,\ \dots,\ x_n\}\backslash\{\alpha : \alpha(x_i) = \prod_{k\neq i} q_{ki}^{j_k} x_i\}$ will be called *type 1*- automorphisms. Those excluded before will be called *type 2*- automorphisms.

2.5 The $\alpha-$derivations

An α-derivation of A is a $\mathbb{C}-$linear map $d_\alpha : \mathbb{C}\{x_1, \ldots, x_n\} \to \mathbb{C}\{x_1, \ldots, x_n\}$, such that

$$d_\alpha(ab) = \alpha(a)d_\alpha(b) + d_\alpha(a)b, \quad \forall a, b \in \mathbb{C}\{x_1, \ldots, x_n\}.$$

Proposition 2.2 *(see [3]) If α is of* type 1*, then every $\alpha-$derivation is inner. If α is of* type 2*, then every $\alpha-$derivation is the sum of an inner derivation and an $\alpha-$central derivation.*

3 The $\alpha-$Hochschild cohomology

3.1 The $\alpha-$Hochschild cohomology in degrees 0 and 1

Redondo and Solotar define in [6] the $\alpha-$Hochschild cohomology of the algebra A with coefficients in the $A-$bimodule M as:

$$HH^n_\alpha(A, M) = H^n(A, \widetilde{M}),$$

where $\widetilde{M}$ is the $A-$bimodule whose structure is given by $a \circ m \circ b = \alpha(a)mb$. In particular, if A is an algebra and M is an A-bimodule, the $\alpha-$Hochschild cohomology of A in degrees zero and one are given by:

$$HH^0_\alpha(A, M) = \{m \in M : ma = \alpha(a)m\}$$

$$HH^1_\alpha(A, M) = Der_\alpha(A, M)/DerInt_\alpha(A, M)$$

For the quantum affine space, we have compute [3]:

Proposition 3.1 *If α is an automorphism of* type 1 *of $\mathbb{C}\{x_1, \ldots, x_n\}$, then*

$$HH^0_\alpha(\mathbb{C}\{x_1, \ldots, x_n\}) = HH^1_\alpha(\mathbb{C}\{x_1, \ldots, x_n\}) = 0.$$

We have also characterized the cohomology groups in degrees zero and one for automorphisms of *type 2* [3].

Proposition 3.2 *Let $\mathbb{C}\{x_1, \ldots, x_n\}$ be the quantun affine $n-$space, $|q_{ij}| > 1, (1 \leq i < j \leq n)$, and α an automorphism of type 2 of $\mathbb{C}\{x_1, \ldots, x_n\}$. Then*

$$HH^0_\alpha(\mathbb{C}\{x_1, \ldots, x_n\}) =< Z^\alpha(\{x^s\}), s \in \mathbb{Z}^n >= \{\lambda x_1^{j_1} \ldots x_n^{j_n}, \lambda \in \mathbb{C}\}$$

and

$$HH^1_\alpha(\mathbb{C}\{x_1, \ldots, x_n\}) = \{d_\alpha : d_\alpha \textit{is an } \alpha-\textit{central derivation of } \mathbb{C}\{x_1, \ldots, x_n\}\}.$$

3.2 The $\alpha-$Hochschild cohomology in degrees 2, 3 and 4

3.2.1 Case $n = 2$

If $A = \mathbb{C}\{x_1, x_2\}$, the $\alpha-$Hochschild cohomology in degree 2 is computed using the Koszul quantum complex defined by Wambst, that is quasi isomorphic to the Hochschild complex , to which the funtor $Hom_{\mathbb{C}}(-,\, S_Q(V)_\alpha)$ is applied , where V is the vector space generated by $x_1, x_2, \ldots, x_n$, Q is the anti-symmetric multiplicatively matrix previously defined and $S_Q(V)_\alpha$ is the algebra of the multiparameter quantum affine space with coefficients in the crossed product by its group of linear automorphisms.

Since $\mathbb{C} \cong \wedge_Q^0(V)$ and $V \cong \wedge_Q^1(V)$, we obtain the complex:

$$0 \to Hom_{\mathbb{C}}(\mathbb{C}, S_Q(V)_\alpha) \xrightarrow{d_0} Hom_{\mathbb{C}}(V, S_Q(V)_\alpha) \xrightarrow{d_1}$$

$$\xrightarrow{d_1} Hom_{\mathbb{C}}(\wedge_Q^2(V), S_Q(V)_\alpha) \xrightarrow{d_2} 0$$

Given $\phi : V \longrightarrow S_Q(V)_\alpha$, such that $\phi = \phi_1 + \phi_2$ with $\phi_1(x_1) = \sum_{i,j} a_{ij} x_1^i x_2^j$, $\phi_1(x_2) = \phi_2(x_1) = 0$, $\phi_2(x_2) = \sum_{i,j} a'_{ij} x_1^i x_2^j$ and $\alpha(x_1) = a_1 x_1$, $\alpha(x_2) = a_2 x_2$, it is straightforward to prove that

$$d_1\phi_1(1.x_1 \wedge x_2) = -(\sum_j (q_{12}a_2 - 1)a_{0j}x_2^{j+1} + \sum_{i>0,j} (q_{21}^{i-1}a_2 - 1)a_{ij}x_1^i x_2^{j+1})$$

and

$$d_1\phi_2(1.x_1 \wedge x_2) = \sum_i (a_1 - q_{12})a'_{i0}x_1^i + \sum_{i,j>0} (a_1 - q_{12}^{1-j})a'_{ij}x_1^{i+1}x_2^j.$$

So, the image of d_1 is spanned by polynomials in x_1, in x_2 and polynomials where x_1x_2 appears.

As $HH_\alpha^2(A) = S_Q(V)_\alpha / Imd_1$, if α is of type 1: $a_1 \neq q_{21}^{j_2}$, $a_2 \neq q_{12}^{j_1}$, then $HH_\alpha^2(A) = \mathbb{C}$.

If α is of type 2, in general $a_1 = q_{21}^{j_2}$, $a_2 = q_{12}^{j_1}$. Therefore:

$$d_1\phi_2(1.x_1 \wedge x_2) = \sum_i (q_{21}^{j_2} - q_{12})a'_{i0}x_1^{i+1} + \sum_{i,j>0} (q_{21}^{j_2} - q_{12}^{1-j})a'_{ij}x_1^{i+1}x_2^j.$$

$$d_1\phi_1(1.x_1 \wedge x_2) = -(\sum_j (q_{12}^{j_1+1} - 1)a_{0j}x_2^{j+1} + \sum_{i>0,j} (q_{21}^{i-1}q_{12}^{j_1} - 1)a_{ij}x_1^i x_2^{j+1}).$$

In general, if $j_1 \neq -1, j_2 \neq -1$, constants and monomials of the form $x_1^{j_1+1}x_2^{j_2+1}$ do not appear in Imd_1, therefore:

$$HH_\alpha^2(A) = \mathbb{C} \oplus \mathbb{C}x_1^{j_1+1}x_2^{j_2+1}.$$

1. If $j_1 = -1, j_2 = -1$, $a_1 = q_{21}^{-1}$, $a_2 = q_{12}^{-1}$, then constants, polynomials in x_1 and polynomials in x_2 do not appear in Imd_1, therefore:

$$HH^2_\alpha(A) = \mathbb{C} \oplus \sum_{i>0} \mathbb{C}x_1^i \oplus \sum_{j>0} \mathbb{C}x_2^j$$

2. If $j_1 \neq -1$ and $j_2 = -1$, $a_1 = q_{21}^{-1}$ and $a_2 = q_{12}^{j_1}$, then constants and polynomials in x_1 do not appear in Imd_1, therefore:

$$HH^2_\alpha(A) = \mathbb{C} \oplus \sum_{i>0} \mathbb{C}x_1^i.$$

3. If $j_1 = -1, j_2 \neq -1$, $a_1 = q_{21}^{j_2}$ and $a_2 = q_{12}^{-1}$, then constants and polynomials in x_2 do not appear in Imd_1, therefore:

$$HH^2_\alpha(A) = \mathbb{C} \oplus \sum_{j>0} \mathbb{C}x_2^j$$

Remark 3.3 *Naidu, Shroff and Witherspoon work [4] with a finite group G acting on $A = S_q(V)$ by graded algebra automorphisms. If G acts diagonally on the basis $\{x_1, x_2, \ldots, x_N\}$ of the vectorial space V, there exit scalars $\lambda_{g,i} \in \mathbb{K}$ such that ${}^g x_i = \lambda_{g,i} x_i$, for all $g \in G$, $i \in \{1, \ldots, N\}$. They describe in the example 4.2 the Hochschild cohomology of $S_q(V) \bowtie G$, for a fix $g \in G$. They consider two special cases. In the first one, q_{12} is not a root of unity and $\lambda_{g,1}$, $\lambda_{g,2}$ are not both equal to 1. They have:*

$$HH^0_g \cong \{0\}\ ;\ HH^1_g \cong \{0\}$$

and
$$HH^2_g \cong span_K\{(1\sharp g) \otimes x_1^* \wedge x_2^*\}.$$

In the second case, q_{12} is simultaneously a primitive $lth-$root of unity, a l_1th-root of $\lambda_{g,1}$ and a l_2th-root of $\lambda_{g,2}^{-1}$.

We are interesting in the first case when q_{12} is not a root of unity and $g \in G$ is a fix element. g acts diagonaly on the basis $\{x_1, x_2\}$ of V in a similar way as the automorphism $\alpha : A \to A$, such that $\alpha(x_i) = a_i x_i$, of type 1. For this type of automorphisms, the $\alpha-$centralizer is null. We obtaine the following results:

$$HH^0_\alpha(\mathbb{C}\{x_1, x_2\}) = HH^1_\alpha(\mathbb{C}\{x_1, x_2\}) = 0$$

and
$$HH^2_\alpha(\mathbb{C}\{x_1, x_2\}) = \mathbb{C},$$

wich are isomorphic to the results obtained by Naidu, Shroff and Witherspoon.

3.2.2 Case n = 3

When $A = \mathbb{C}\{x_1, x_2, x_3\}$ and if α is of type 1, we obtain that $HH^2_\alpha(A) = 0$ and $HH^3_\alpha(A) = \mathbb{C}$.

If α is of type 2, there are $j_1, j_2, j_3 \in \mathbb{Z}$ such that $a_1 = q_{21}^{j_2} q_{31}^{j_3}$, $a_2 = q_{12}^{j_1} q_{32}^{j_2}$, $a_3 = q_{13}^{j_1} q_{23}^{j_2}$.

Here, we consider those automorphisms for which the exponents j_1, j_2, j_3 don't make the corresponding powers of q_{13}, q_{21} and q_{23} equal.

In general, $HH^3_\alpha(A) = \mathbb{C} \oplus \mathbb{C} x_1^{j_1+1} x_2^{j_2+1} x_3^{j_3+1}$ if j_1, j_2, j_3 are not equal to -1 . Next we analyze some particular cases:

1. If only $j_3 = j_2 = -1$, we have:
$$HH^3_\alpha(A) = \mathbb{C} \oplus \sum_{i>0} \mathbb{C} x_1^i.$$
2. On the other hand, if $j_1 = j_3 = -1$, we get:
$$HH^3_\alpha(A) = \mathbb{C} \oplus \sum_{j>0} \mathbb{C} x_2^j.$$
3. If $j_1 = j_2 = -1$:
$$HH^3_\alpha(A) = \mathbb{C} \oplus \sum_{k>0} \mathbb{C} x_3^k.$$
4. If $j_1 = j_2 = j_3 = -1$:
$$HH^3_\alpha(A) = \mathbb{C} \oplus \sum_{i>0} \mathbb{C} x_1^i \oplus \sum_{j>0} \mathbb{C} x_2^j \oplus \sum_{k>0} \mathbb{C} x_3^k.$$

3.2.3 Case $n = 4$

For $A = \mathbb{C}\{x_1, x_2, x_3, x_4\}$, we first consider, as above, the case such that there are no $m \in \mathbb{Z}^4$, $m \neq 0$, with $(q_{32})^{m_3} = (q_{21})^{m_1}, (q_{32})^{m_3} = (q_{41})^{m_4}$.

If α is of type 1, $HH^2_\alpha(A) = HH^3_\alpha(A) = 0$ and $HH^4_\alpha(A) = \mathbb{C}$.

When α is of type 2, there are $j_1, j_2, j_3, j_4 \in \mathbb{Z}$ such that $\alpha(x_1) = q_{21}^{j_2} q_{31}^{j_3} q_{41}^{j_4} x_1$, $\alpha(x_2) = q_{12}^{j_1} q_{32}^{j_3} q_{42}^{j_4} x_2$, $\alpha(x_3) = q_{13}^{j_1} q_{23}^{j_2} q_{43}^{j_4} x_3$ and $\alpha(x_4) = q_{14}^{j_1} q_{24}^{j_2} q_{34}^{j_3} x_4$; then

$HH^4_\alpha(A) = \mathbb{C} \bigoplus \mathbb{C} x_1^{j_1+1} x_2^{j_2+1} x_3^{j_3+1} x_4^{j_4+1}$ if $j_1 \neq -1$, $j_2 \neq -1$, $j_3 \neq -1$ and $j_4 \neq -1$.

But if

1. $j_2 = j_3 = j_4 = -1$, then $HH^4_\alpha(A) = \mathbb{C} \oplus \sum \mathbb{C} x_1^i$.
2. $j_1 = j_3 = j_4 = -1$, then $HH^4_\alpha(A) = \mathbb{C} \oplus \sum \mathbb{C} x_2^j$.
3. $j_1 = j_2 = j_4 = -1$,then $HH^4_\alpha(A) = \mathbb{C} \oplus \sum \mathbb{C} x_3^k$.

4. $j_1 = j_2 = j_3 = -1$, then $HH^4_\alpha(A) = \mathbb{C} \oplus \sum \mathbb{C}x_4^l$.

5. $j_1 = j_2 = j_3 = j_4 = -1$, then $HH^4_\alpha(A) = \mathbb{C} \oplus \sum \mathbb{C}x_1^i \oplus \sum \mathbb{C}x_2^j \oplus \sum \mathbb{C}x_3^k \oplus \sum \mathbb{C}x_4^l$.

6. For any value of j_1 and j_2 (different from -1) and $j_3 = j_4 = -1$,then $HH^4_\alpha(A) = \mathbb{C} \oplus \mathbb{C}x_1^{j_1+1}x_2^{j_2+1}$.

7. For any value of j_1 and j_3 (different from -1) and $j_2 = j_4 = -1$,then $HH^4_\alpha(A) = \mathbb{C} \oplus \mathbb{C}x_1^{j_1+1}x_3^{j_3+1}$.

4 α-Hochschild homology of the multiparameter quantum affine space

We use some techniques defined by Wambst [12] for the computation of the α-Hochschild homology of the quantum algebra A_α. In this case we need to define:

$$C = \{\gamma \in \mathbb{N}^n / \text{ for every } i, 1 \leq i \leq n, \gamma_i = 0 \text{ or } x^\gamma x_i = \alpha(x_i)x^\gamma\}.$$

Theorem 4.1 *Let $A = \mathbb{C}\{x_1, \ldots, x_n\}$ be the $\mathbb{C}$-algebra with relations $x_i x_j = q_{ij} x_j x_i$, $1 \leq i, j \leq n$ and the linear automorphism α of A with $\alpha(x_i) = a_i x_i$, $1 \leq i \leq n$. Then the α-Hochschild homology groups of the multiparameter quantum algebra A on $\mathbb{C}$ are given by:*

$$(HH_\alpha)_m(A) = \bigoplus_{\substack{\beta \in \{0,1\}^n \\ |\beta| = m}} \bigoplus_{\substack{\gamma \in \mathbb{N}^n \\ \beta + \gamma \in C}} \mathbb{C}x^\gamma \otimes x^\beta$$

for all $m \in \mathbb{N}$.

<u>Proof</u>

According to Wambst's definition of Koszul complexes [12], the α-Hochschild homology groups of A are isomorphic to the homology groups of the complex $K_*(A, A_\alpha)$, where A_α is as before. The differential of the complex is:

$$\widehat{d}(x^\gamma \otimes x_{i_1} \wedge \ldots \wedge x_{i_p}) = \sum_{k=1}^{p} (-1)^{k+1} [\prod_{s=1}^{k-1} q_{i_s i_k} x^\gamma x_{i_k} - \prod_{s=k+1}^{p} q_{i_k i_s} \alpha(x_{i_k}) x^\gamma]$$

$$\otimes x_{i_1} \wedge \ldots \wedge \widehat{x}_{i_k} \wedge \ldots \wedge x_{i_p}$$

on a homogeneous element of $S \otimes \Lambda$, where $S = S_Q(V)$ and $\Lambda = \Lambda_Q(V)$, that is to say, the algebra of the multiparameter quantum affine space V and the analog of the exterior algebra, linearly generated by the family $\{x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}, \varepsilon_i \in \{0,1\}\}$, respectively.

The homogeneous element $x_{i_1} \wedge \dots \wedge x_{i_p}$ of Λ, is indicated by x^β, with β the multi-integer such that $\beta_{i_k} = 1$ if k varies from 1 to p and $\beta_j = 0$ in any other case. Then the differential $\widehat{d}$ is:

$$\widehat{d}(x^\gamma \otimes x^\beta) = \sum_{i=1}^{n} \Omega(\gamma, \beta, i) x^{\gamma+[i]} \otimes x^{\beta-[i]}$$

where the value of $\Omega(\gamma, \beta, i)$ is null if $x^{\gamma+\beta} x_i = \alpha(x_i) x^{\gamma+\beta}$ or if $\beta_i = 0$ and, in any other case, it is $\varepsilon(\beta, i)(\prod_{s=1}^{k-1} q_{sk}^{\beta_s} \prod_{h=k+1}^{n} q_{hk}^{\gamma_h} - \prod_{s=k+1}^{n} q_{ks}^{\beta_s} a_k \prod_{l=1}^{k-1} q_{kl}^{\gamma_l})$, with $\varepsilon(\beta, i) = (-1)^{\sum_{s=1}^{i-1} \beta_s}$, $\gamma \in \mathbb{N}^n$ and $\beta \in \{0,1\}^n$, $i = 1, \dots, n$.

The complex $K_*(A, A_\alpha)$ is graded with $deg(x^\gamma \otimes x^\beta) = \gamma + \beta$. Given $\delta \in \mathbb{N}^n$, let K_δ be the subcomplex of elements of degree δ. It is possible to consider these subcomplexes since the image through $\widehat{d}$ of an homogeneous element of degree δ is also of degree δ. If $\delta \in C$, the restriction of $\widehat{d}$ to K_δ is null. If $\delta \notin C$, it will be shown that K_δ is acyclic using an homotopy $\widetilde{h}$. So $HH_\alpha(A) = \bigoplus_{\delta \in C} K_\delta$.

The homotopy $\widetilde{h}$ is defined as:

$$\widetilde{h}(x^\gamma \otimes x^\beta) = \frac{1}{\|\gamma + \beta\|} \sum_{i=1}^{n} \omega(\gamma, \beta, i) x^{\gamma-[i]} \otimes x^{\beta+[i]}$$

where $\omega(\gamma, \beta, i)$ is null if $x^{\gamma+\beta} x_i = \alpha(x_i) x^{\gamma+\beta}$, if $\beta_i = 1$ or if $\gamma_i = 0$ and in any other case it is equal to $[\Omega(\gamma - [i], \beta + [i], i)]^{-1}$, with $\gamma \in \mathbb{N}^n$ and $\beta \in \{0,1\}^n$, such that $\gamma + \beta \notin C$ and $i = 1, \dots, n$. On the other hand, for $\gamma \in \mathbb{N}^n$, $\|\gamma\|$ denotes the number of integers i, $1 \leq i \leq n$, such that $\alpha(x_i) x^\gamma \neq x^\gamma x_i$ and $\gamma_i \neq 0$. It is well defined, since if $\gamma \notin C$, then $\|\gamma\| \neq 0$.

Then, $\widetilde{h}$ is an endomorphism of degree 1 of K_γ. We will prove that it verifies

$$\widetilde{h} \circ \widehat{d} + \widehat{d} \circ \widetilde{h} = id.$$

For this, we shall compute $\widetilde{h} \circ \widehat{d} + \widehat{d} \circ \widetilde{h}$ on an homogeneous element $x^\gamma \otimes x^\beta$:

$$(\widetilde{h} \circ \widehat{d} + \widehat{d} \circ \widetilde{h})(x^\gamma \otimes x^\beta) =$$

$$= \frac{1}{\|\gamma + \beta\|} \sum_{i=1}^{n} [\Omega(\gamma, \beta, i) \omega(\gamma + [i], \beta - [i], i) + \omega(\gamma, \beta, i) \Omega(\gamma - [i], \beta + [i], i)] x^\gamma \otimes x^\beta +$$

$$+ \frac{1}{\|\gamma + \beta\|} \sum_{i \neq j} [\omega(\gamma, \beta, j) \Omega(\gamma - [j], \beta + [j], i) + \Omega(\gamma, \beta, i) \omega(\gamma + [i], \beta - [i], j)] \times$$

$$x^{\gamma-[j]+[i]} \otimes x^{\beta+[j]-[i]}$$

When $i \neq j$:

$$\omega(\gamma,\beta,j)\Omega(\gamma-[j],\beta+[j],i)+\Omega(\gamma,\beta,i)\omega(\gamma+[i],\beta-[i],j)=0.$$

Indeed, let $\Gamma(i,j)=\omega(\gamma,\beta,j)\Omega(\gamma-[j],\beta+[j],i)$ and $\Gamma'(i,j)=\Omega(\gamma,\beta,i)\omega(\gamma+[i],\beta-[i],j)$.

We will see that either $\Gamma(i,j)=\Gamma'(i,j)=0$ or they are both not null.

If $\Gamma(i,j)=0$, then $\omega(\gamma,\beta,j)=0$ or $\Omega(\gamma-[j],\beta+[j],i)=0$. For $\omega(\gamma,\beta,j)$ to be zero one must have either $x^{\gamma+\beta}x_j=\alpha(x_j)x^{\gamma+\beta}$, or $\beta_j=1$, or $\gamma_j=0$. In the first case, $x^{\gamma+[i]+\beta-[i]}x_j=x_jx^{\gamma+[i]+\beta-[i]}$ and therefore $\omega(\gamma+[i],\beta-[i],j)=0$ and $\Gamma'(i,j)=0$. In the second case, $\beta_j=1$ and then $(\beta-[i])_j=\beta_j=1$, which implies that $\Gamma'(i,j)=0$. In the third one $\gamma_j=0$, so $(\gamma+[i])_j=1$ and then $\omega(\gamma+[i],\beta-[i],j)=0$, so $\Gamma'(i,j)=0$.

For $\Omega(\gamma-[j],\beta+[j],i)$ to be zero one must have either $x^{\gamma-[j]+\beta+[j]}x_i=\alpha(x_i)x^{\gamma+\beta}$, or $\beta_{j+1}=0$. In the first case, $x^{\gamma-[j]+\beta+[j]}x_i=\alpha(x_i)x^{\gamma+\beta}$ and therefore $\Omega(\gamma,\beta,i)=0$ and $\Gamma'(i,j)=0$. In the second case, $\beta_j+1=0$ and then $\beta_j=(\beta-[i])_j=1$, which implies that $\omega(\gamma+[i],\beta-[i],j)=0$ and therefore $\Gamma'(i,j)=0$.

It may be prove in a similar way that $\Gamma'(i,j)=0$ implies that $\Gamma(i,j)=0$.

We study the cases $i<j$ and $i>j$. Let us suppose that $i<j$. In this case

$$\begin{aligned}
\Gamma(i,j) &= \omega(\gamma,\beta,j)\Omega(\gamma-[j],\beta+[j],i)\\
&= [\Omega(\gamma-[j],\beta+[j],j)]^{-1}\Omega(\gamma-[j],\beta+[j],i)\\
&= (\varepsilon(\beta-[j],j))^{-1}(\textstyle\prod_{s=1}^{j}q_{sk}^{\beta_s}\prod_{r=j+1}^{n}q_{jr}^{-\gamma_r}-\prod_{s=j+1}^{n}q_{js}^{\beta_s}a_j\prod_{r=1}^{j-1}q_{rj}^{-\gamma_r})^{-1}\times\\
&\quad \varepsilon(\beta+[j],i)(\textstyle\prod_{s=1}^{i}q_{si}^{\beta_s}\prod_{r=i+1}^{n}q_{ir}^{-\gamma_r}q_{ij}-\prod_{s=i+1}^{n}q_{is}^{\beta_s}q_{ij}a_r\prod_{r=1}^{i-1}q_{ri}^{-\gamma_r})\\
\Gamma'(i,j) &= \Omega(\gamma,\beta,i)\omega(\gamma+[i],\beta-[i],j)\\
&= \varepsilon(\beta,i)(\textstyle\prod_{s=1}^{i}q_{si}^{\beta_s}\prod_{r=i+1}^{n}q_{ir}^{-\gamma_r}-\prod_{s=i+1}^{n}q_{is}^{\beta_s}a_i\prod_{r=1}^{i-1}q_{ri}^{-\gamma_r})\times\\
&\quad (\varepsilon(\beta+[i]-[j],j))^{-1}(\textstyle\prod_{s=1}^{j}q_{sj}^{\beta_s}q_{ji}\prod_{r=j+1}^{n}q_{jr}^{-\gamma_r}-\\
&\quad \textstyle\prod_{s=j+1}^{n}q_{js}^{\beta_s}a_j\prod_{r=1}^{j-1}q_{rj}^{-\gamma_r}q_{ji})^{-1}
\end{aligned}$$

Evidently,

$$\frac{\prod_{s=1}^{i}q_{si}^{\beta_s}\prod_{r=i+1}^{n}q_{ir}^{-\gamma_r}q_{ij}-\prod_{s=i+1}^{n}q_{is}^{\beta_s}q_{ij}a_r\prod_{r=1}^{i-1}q_{ri}^{-\gamma_r}}{\prod_{s=1}^{j}q_{sj}^{\beta_s}q_{ji}\prod_{r=j+1}^{n}q_{jr}^{-\gamma_r}-\prod_{s=j+1}^{n}q_{js}^{\beta_s}a_j\prod_{r=1}^{j-1}q_{rj}^{-\gamma_r}q_{ji}}=$$

$$=\frac{\prod_{s=1}^{i}q_{si}^{\beta_s}\prod_{r=i+1}^{n}q_{ir}^{-\gamma_r}-\prod_{s=i+1}^{n}q_{is}^{\beta_s}a_i\prod_{r=1}^{i-1}q_{ri}^{-\gamma_r}}{\prod_{s=1}^{j}q_{sj}^{\beta_s}q_{ji}\prod_{r=j+1}^{n}q_{jr}^{-\gamma_r}-\prod_{s=j+1}^{n}q_{js}^{\beta_s}a_j\prod_{r=1}^{j-1}q_{rj}^{-\gamma_r}q_{ji}}.$$

Besides, if $i<j$:

$$\varepsilon(\beta-[j],j)\varepsilon(\beta+[j],j)=(-1)^{\beta_1+\beta_2+\ldots+\beta_{j-1}}(-1)^{\beta_1+\beta_2+\ldots+\beta_{i-1}}$$

and

$$\varepsilon(\beta,i)\varepsilon(\beta+[i]-[j],j)=(-1)^{\beta_1+\beta_2+\ldots+\beta_{i-1}}(-1)^{\beta_1+\ldots+(\beta_i+1)+\ldots+\beta_{j-1}}.$$

Both expressions have opposite signs. The case $i>j$ is analogous. As a consequence:

$$\sum_{i=1}^{n}(\omega(\gamma,\beta,i)\Omega(\gamma-[i],\beta+[i],i)+\Omega(\gamma,\beta,i)\omega(\gamma+[i],\beta-[i],i))=\|\gamma+\beta\|.$$

Since $\gamma\notin C$, $\gamma+\beta\neq 0$. We denote $\omega_1=\omega(\gamma,\beta,i)$, $\Omega_1=\Omega(\gamma-[i],\beta+[i],i)$, $\omega_2=\omega(\gamma+[i],\beta-[i],i)$ and $\Omega_2=\Omega(\gamma,\beta,i)$. If $\gamma_i=0$ and $\beta_i\neq 0$, then $\omega_1=\Omega_2$ and if $\gamma_i\neq 0$ and $\beta_i=0$, then $\omega_1\Omega_1=1$ and $\omega_2\Omega_2=0$. If $\gamma_i\neq 0$ and $\beta_i\neq 0$, then $\omega_1\Omega_1=0$ and $\omega_2\Omega_2=1$.

From the previous equality, we obtain the following:

$$(\widetilde{h}\circ\widehat{d}+\widehat{d}\circ\widetilde{h})(x^\gamma\otimes x^\beta)=\frac{\|\gamma+\beta\|}{\|\gamma+\beta\|}x^\gamma\otimes x^\beta=x^\gamma\otimes x^\beta.\square$$

Corollary 4.2 *For α of* type 1*, the $\alpha-$centralizer is empty, so $C=\{(0,0,\ldots,0)\}$ .Therefore:*

$(HH_\alpha)_0(A)=\mathbb{C}$ *and* $(HH_\alpha)_m(A)=0, m>0$.

Remark 4.3 *Richard [8] studies a family of iterated Ore extensions called twisted polynomial algebras $S^\Lambda_{n,r}(k)$, which are parametrized by an integer $n\geq 1$, an integer r, with $0\leq r\leq n$ and a multiplicatively antisymmetric matrix $\Lambda=(\lambda_{ij})_{1\leq i,j\leq n}\in M_n(K^*)$. He computes their homologies and obtain that the Hochschild homology of $S^\Lambda_{n,r}(k)$ is null in degrees greater than $n+r$. He observes that for $r=0$, the algebra $S^\Lambda_{n,0}(k)$ is the quantum affine space $\mathcal{O}_\Lambda(k^n)$.*

If we restrict our results for the case in which the automorphism α (of type 1) is the identity, we reobtain his results.

Corollary 4.4 *For α of* type 2*, the α-centralizer is $\{x_1^{j_1},\ldots,x_n^{j_n}\}$, then $C=\{(0,0,\ldots,0),(j_1,j_2,\ldots,j_n)\}$. As a consequence,*

$$\begin{array}{rcl}(HH_\alpha)_0(A) & = & \mathbb{C}\oplus\mathbb{C}x_1^{j_1}\ldots x_n^{j_n}\otimes 1\\ (HH_\alpha)_1(A) & = & \bigoplus_{i=1}^n\mathbb{C}x_1^{j_1}\ldots x_i^{j_i-1}\ldots x_n^{j_n}\otimes x_i\\ (HH_\alpha)_2(A) & = & \bigoplus_{i<k}\mathbb{C}x_1^{j_1}\ldots x_i^{j_i-1}\ldots x_k^{j_k-1}\ldots x_n^{j_n}\otimes x_ix_k\\ (HH_\alpha)_3(A) & = & \bigoplus_{i<k<h}\mathbb{C}x_1^{j_1}\ldots x_i^{j_i-1}\ldots x_k^{j_k-1}\ldots x_h^{j_h-1}\ldots x_n^{j_n}\otimes x_ix_kx_h\end{array}$$

And finally:

$$(HH_\alpha)_n(A)=\mathbb{C}x_1^{j_1}x_2^{j_2-1}\ldots x_n^{j_n-1}\otimes x_1x_2\ldots x_n.$$

5 Relation between α-Hochschild homology and α-Hochschild cohomology of A

Several authors, as Michel Van den Bergh, Roland Berger and Nicolas Marconnet, have studied the relationship between Hochschild homology and cohomology for different kind of rings and modules. Berger and Marconnet [1] proved, based on Van den Bergh's duality theorem, that for an algebra A, N-homogeneous that is AS-Gorestein and of finite global dimension D for any bimodule M, it is verified that

$$HH^i(A, M) \cong HH_{D-i}(A, {}_{\varepsilon^{D+1}\phi} M)$$

Theorem 5.1 *Let $A = \mathbb{C}\{x_1, \ldots, x_n\}$ be the $\mathbb{C}-$algebra with the relations $x_i x_j = q_{ij} x_j x_i$ for $1 \leq i < j \leq n$ and α the linear automorphism of A with $\alpha(x_i) = a_i x_i$. Then the relations between the $\alpha-$Hochschild homology and the $\alpha-$Hochschild cohomology of A are*

$$(HH_\alpha)^i(A) \cong (HH_{\varphi^{-1}\alpha})_{(n-i)}(A)$$

and

$$(HH_{\varphi\alpha})^i(A) \cong (HH_\alpha)_{(n-i)}(A)$$

The algebra $A = \mathbb{C}\{x_1, \ldots, x_n\}$, belongs to the class of the N-homogeneous graded algebra, when $N = 2$. It is also known [9] that this algebra has $\{x_1, \ldots, x_n\}$ as PBW base, which is an enough condition for A to be a Koszul homogeneous algebra,just as Priddy has proved [5] it. Berger and Marconnet [1], in their theorem 4.5, prove that for a Koszul homogeneous algebra, the Hochschild dimension and global dimension are equal. In our case, the Hochschild dimension of A is n and therefore the $gldimA = n$ is obviously finite. Besides, A is AS-regular and consequently is AS-Gorestein. Van den Bergh shows in his proposition 2 [10], the Frobenius automorphism of $A!$, which he calls $\phi!$, and the automorphism ϕ of A which is the adjoint of $\phi!$. On the other hand, ε is the automorphism of A which is the product by $(-1)^m$ on the homogeneous component of degree m of A. For the case $N = 2$, the algebra of Yoneda of A is always symmetric graded and so results $\phi = \varepsilon^{n+1}$ and then $\varepsilon^{n+1}\phi = 1_A$.

${}_{\varepsilon^{n+1}\phi}M$ in this case is:

$${}_{\varepsilon^{n+1}\phi}A_\alpha \cong U \otimes_A A_\alpha$$

where $U = Ext^i_{A^e}(A, A^e)$ –by proposition 2 of [10]–, that is 0 if $i \neq n$ ([1], section 6) and is ${}_{\varepsilon^{n+1}\phi}A_\alpha$, it is to say ${}_\varphi A_\alpha$, if $i = n$. φ is an automorphism of A defined by $\varphi = 1_A$ if α is of *type 1* and $\varphi(x_i) = \prod_{k \neq i} q_{ik} x_i$ if α is of *type 2*.

Then:

$$(HH_\alpha)^i(A) \cong (HH_{\varphi^{-1}\alpha})_{(n-i)}(A)$$

and

$$(HH_{\varphi\alpha})^i(A) \cong (HH_\alpha)_{(n-i)}(A).$$

Remark 5.2 *Computing $(HH_{\varphi^{-1}\alpha})_n(A)$, we obtain for $n = 2$:*

$$(HH_{\varphi^{-1}\alpha})_0(A) = \mathbb{C} \oplus \mathbb{C}x_1^{j_1+1}x_2^{j_2+1} \otimes 1$$

$$(HH_{\varphi^{-1}\alpha})_2(A) = \mathbb{C}x_1^{j_1}x_2^{j_2} \otimes x_1x_2;$$

for $n = 3$:

$$(HH_{\varphi^{-1}\alpha})_0(A) = \mathbb{C} \oplus \mathbb{C}x_1^{j_1+1}x_2^{j_2+1}x_3^{j_3+1} \otimes 1$$

$$(HH_{\varphi^{-1}\alpha})_3(A) = \mathbb{C}x_1^{j_1}x_2^{j_2}x_3^{j_3} \otimes x_1x_2x_3$$

that are the same results that we obtained for direct computations.

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Received: April, 2012