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On the Prime Spectrum of a Module over Noncommutative Rings

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Abstract. Let R be an associative ring with identity and M an R-module. Let $Spec\,(M)$ be the set of all prime submodules of M. We topologize $Spec\,(M)$ with the Zariski topology and prove some useful results.

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In this paper, we always assume that a ring is associative with identity, and by an ideal we mean a 2-sided ideal.Let R be a associative ring with identity and M be a left R-module.By a prime submodule(or a p-prime submodule)of M, we mean a proper submodule P with $(P : M) = \{r \in R : rM \subseteq P\} = p$ such that $rRm \subseteq P$ for $r \in R$ and $m \in M$ implies that either $m \in P$ or $r \in p$. Recall that a proper ideal P of a ring R is called prime if $aRb \subseteq P$ implies that either $a \in P$ or $b \in P$.It is clear that if N is a prime submodule, then $(N : M) = \{r \in R : rM \subseteq N\}$ is a prime ideal of R. The set of all prime submodules of M is called the prime spectrum of M and denoted by $Spec\,(M)$ or X^M. Several authors have extended the notation of prime ideals to modules(see,for example, $[1], [5] - [8], [11]$).Note that $Spec\,(M)$ may be empty (see, 7). Throughout this paper we assume that $Spec\,(M)$ is not empty.We introduce a topology called the Zariski topology on $X^M = Spec\,(M)$ for any R-module, in which closed sets are varieties $V\,(N) = \left\{P \in X^M : N \subseteq P\right\}$ where N is a subset of an R-module M. Clearly,

$V(E) = V(S)$ where S is an $R-$submodule of M generated by a subset E of M. We write $N \leq M$ to indicate that N is a submodule of M.

Recall that a ring R is called prime if (0) is prime ideal of R.

Definition 1. *Let M be an R module. Then a proper submodule N of M is reducible if it can be written as the intersection $N = S_1 \cap S_2$ of two submodules S_1, S_2 with $N \neq S_1$ and $N \neq S_2$, otherwise N is irreducible.*

Proposition 1. *For any subset E of M, we consider varieties denoted by $V(E)$. We define $V(E) = \{P \in Spec(M) : E \subseteq P\}$. Then*

(a) If N is a submodule generated by E, then $V^R(E) = V^R(N)$.
(b) $V(0_M) = Spec(M)$ and $V(M) = \varnothing$.
(c) $\bigcap_{i \in J} V(N_i) = V\left(\sum_{i \in J} N_i\right)$ for any index set J.
(d) $V(N) \cup V(L) \subseteq V(N \cap L)$, where $N, L \leq M$.

Proof. [see, [2] − [4]] .

Our purpose is to study modules for which the inclusion of (d) proposition 1 is always an equality.These modules called Top-module.

An R-module M is called a multiplication module if for each $N \leq M$, there exists an ideal $I \leq R$ such that $N = IM$. Then, $N = (N : M)M$. Indeed, $N = IM \subseteq (IM : M)M = (N : M)M \subseteq N$. If N is a prime submodule of a multiplication R-module M, then $N_1 \cap N_2 \subseteq N$ implies $N_1 \subseteq N$ or $N_2 \subseteq N$ where $N_1, N_2 \leq M$ [see , for more detail, [9] − [10]] .

Proposition 2. *Every multiplication module is a Top- module.*

Proof. Let $N \in V(N_1 \cap N_2)$ and so $N_1 \cap N_2 \subseteq N$ .Then , $N_1 \subseteq N$ or $N_2 \subseteq N$. Therefore $N \in V(N_1)$ or $N \in V(N_2)$ ∎

Definition 2. *Let M be an R-module. For every subset Y of X^M, let us denote by $J(Y)$ the intersection of all prime submodules of M which belong to Y.*

Definition 3. *Let M be an R module.M is distributive if it satisfy the following condition $(S_1+S_2)\cap N = (S_1\cap N)+(S_2\cap N)$ for all submodules S_1, S_2 and N of M.*

For any submodule N of an $R-$module$M$,the radical, $radN$, of N is defined to be the intersection of all prime submodules of M containing N, and in case N is not contained in any prime submodule then $radN$ is defined to be M. The radical of the module M is defined to be $rad(0)$.

Theorem 1. *M is a Top-module and $radS = S$ for each submodule S of M.Then M is a distributive module.*

Proof. Let S_1, S_2 and N be any submodules of M.Then,
$(S_1 + S_2) \cap N = rad((S_1 + S_2) \cap N)$

$= J(V((S_1 + S_2) \cap N))$

$= J(V(S_1 + S_2) \cup V(N))$

$= J(V(S_1 \cup S_2) \cup V(N))$

$= J((V(S_1) \cap V(S_2)) \cup V(N))$

$= J((V(S_1) \cup V(N)) \cap (V(S_2) \cup V(N)))$

$= J(V(S_1 \cap N) \cap V(S_2 \cap N))$

$= J(V((S_1 \cap N) \cup (S_2 \cap N)))$

$= J(V((S_1 \cap N) + (S_2 \cap N)))$

$= rad((S_1 \cap N) + (S_2 \cap N))$

$= (S_1 \cap N) + (S_2 \cap N)$

∎

Lemma 1. *Let M be a Top-module and N_1, N_2 be two submodules of M .Then the equalities $rad(N_1 \cap N_2) = radN_1 \cap radN_2$ holds.*

Proof. $rad(N_1 \cap N_2) = J(V(N_1 \cap N_2))$

$= J(V(N_1) \cup V(N_2))$

$= J(V(N_1) \cap J(V(N_2))$

$= radN_1 \cap radN_2$ ∎

Note that the closure of a subset Y of $Spec(M)$ denoted by $\overline{Y}$.

Theorem 2. *Let M be a Top-module. Then $\overline{Y} = V((J(Y))$.*

Proof. Let $V(S)$ be a closed set containing Y. Then $S \subseteq N$ for every prime submodule N in Y, so $S \subseteq J(Y)$ and consequently $V(J(Y)) \subseteq V(S)$.Since $Y \subseteq V(J(Y))$, then $V(J(Y))$ is the smallest closed subset of X^M containing Y. Thus $\overline{Y} = V((J(Y))$. ∎

A topological space X is T_1-space if and only if given any two distinct points x and y in X , each lies in an open sets which does not contain the other.

Theorem 3. *X^M is T_1-space if and only if each prime submodule is maximal in the family of all prime submodule of M.*

Proof. Suppose N is maximal in $Spec(M)$.Then $\left\{\overline{N}\right\} = V(J(N)) = V(N)$ and since N is maximal submodule so $\left\{\overline{N}\right\} = \{N\}$, this means $\{N\}$ is closed.Then X^M is a $T_1 - space$, and vice versa. ∎

Definition 4. *A topological space X is called irreducible if every finite intersection of non-empty open sets of X is non-empty.*

Proposition 3. *Let M be a Top-module and Y a subset of X^M .If $J(Y)$ is prime submodule, then Y is an irreducible space.*

Proof. Suppose $N = J(Y)$ prime submodule. $\overline{Y} = V(J(Y)) = V(N)$ by Theorem 2. $\overline{Y} = V(N) = V(J(N)) = \left\{\overline{N}\right\}$. As a set consisting of a single element is irreducible ,then $\left\{\overline{N}\right\}$ is irreducible, that is $\overline{Y}$ is irreducible .Then Y is irreducible. ∎

Corollary 1. *Let M be a Top-module.Then $V(N)$ is an irreducible space for every prime submodule N.*

Proof. Since $J(V(N)) = \bigcap\limits_{N \subseteq P} P = radN = N$, $V(N)$ is irreducible space by proposition 3. ∎

We denote the complement of $V(N)$ by $D(N)$. Note that $D(m) = D(Rm)$ for every $m \in M$.

Theorem 4. *Let M be a Top-module.Then the sets $D(m_i)\,(i \in I)$ form a base of X^M.*

Proof. Let $D(S)$ be an open set, where S is a submodule of M which is in the form $S = \underset{i\in I}{\cup}\{m_i\}$,$m_i \in S$, then $D(S) = D(\underset{i\in I}{\cup}\{m_i\}) = \underset{i\in I}{\cup} D(m_i)$. ∎

Theorem 5. *Let M be a Noetherian Top-module. Then each open set of X^M is compact.*

Proof. Suppose $D(S)$ is an open set of X. Let $\{D(m_i)\}_{i\in I}$ be a basic open cover, $m_i \in M$, for each $i \in I$.

$D(S) \subseteq \underset{i\in I}{\cup} D(m_i) = D(\underset{i\in I}{\cup} m_i)$ so $D(S) \subseteq D(K)$ where K is the submodule of M generated by $A = \{m_i\}_{i\in I}$. Since M is Noetherian, K is finitely generated.Let $K =< b_1, ..., b_r >$ .Thus, $b_i = \sum_{j=1}^{n} r_{ij} m_{ij}$ where $m_{ij} \in A$.That is there exists $\{m_{i1}, ..., m_{in}\} \subseteq A$ such that $K =< m_{i1}, ..., m_{in} >$. Then $D(S) \subseteq D(< m_{i1}, ..., m_{in} >)$ so $D(S) \subseteq \overset{n}{\underset{i=1}{\cup}} D(m_i)$.Thus $D(S)$ is compact. ∎

Proposition 4. *Let M be a Top- module such that every open set of $spec(M)$ is compact and $radS = S$ for each submodule S of M. Then M is Noetherian module.*

Proof. Let $S_1 \subseteq S_2 \subseteq ... \subseteq S_n \subseteq ...$ be an ascending chain of submodules of M.Then $D(S_1) \subseteq D(S_2) \subseteq .... \subseteq ...D(S_n) \subseteq ...$. Let $K = \underset{i\in I}{\cup} S_i$.Then $D(K) = D(\underset{i\in I}{\cup} S_i) = \underset{i\in I}{\cup} D(S_i)$.Thus by Theorem 5,.$D(K) = \overset{n}{\underset{i=1}{\cup}} D(S_i)$ that is $D(K) = D(\overset{n}{\underset{i=1}{\cup}} S_i)$ so $V(K) = V(\overset{n}{\underset{i=1}{\cup}} S_i)$. Hence, $J(V(K)) = J(V(\overset{n}{\underset{i=1}{\cup}} S_i))$.Therefore $radK = rad(\overset{n}{\underset{i=1}{\cup}} S_i)$ and by hypothesis $K = \overset{n}{\underset{i=1}{\cup}} S_i$.Hence $K = S_j$ for some $j \in I$.Thus M is Noetherian. ∎

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